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Abstract

In this article we are interested in the rigorous construction of geometric optics expansions for hyperbolic corner problems. More precisely we focus on the case where selfinteracting phases occur. Those phases are proper to the high frequency asymptotics for the corner problem and correspond to rays that can display a homothetic pattern after a suitable number of reflections on the boundary. To construct the geometric optics expansions in that framework, it is necessary to solve a new amplitude equation in view of initializing the resolution of the WKB cascade.

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1 Introduction.

The aim of this article is to give rigorous methods to construct geometric optics expansions for linear hyperbolic initial boundary value problems in the quarter space. Such problems will be called corner problems and read :

$$\begin{cases} L(\partial)u^\varepsilon := \partial_t u^\varepsilon + A_1\partial_1 u^\varepsilon + A_2\partial_2 u^\varepsilon = 0, \quad (x_1,x_2)\in\mathbb{R}_+\times\mathbb{R}_+,\ t\geq 0, \\ B_1 u^\varepsilon_{|x_1=0} = g^\varepsilon, \\ B_2 u^\varepsilon_{|x_2=0} = 0, \\ u^\varepsilon_{|t\leq 0} = 0, \end{cases} \tag{1}$$

where the matrices A_1, A_2 are in $M_N(\mathbb{R})$ and where the boundary matrices B_1, B_2 are elements of $M_{p_1,N}(\mathbb{R})$ and $M_{p_2,N}(\mathbb{R})$ respectively (the values of the integers p_1 and p_2 will be made precise in Assumption 2.2).

We have, in this article, chosen to work with only two space dimensions in order to save some notations. However, all the following results can be generalized if one looks at problem (1) with extra space variables $x' \in \mathbb{R}^{d-2}$ (with, of course, the suitable modifications on the operator $L(\partial)$ to preserve hyperbolicity).

This article can be seen, in some sense, as a complement to the paper by Sarason and Smoller [17] in which the authors give intuitions and elements of proof about how to construct geometric optics expansions but where the construction is not performed rigorously. To our knowledge it is the only paper about this subject in the literature for general first order systems and we shall rely on some of the deep ideas of this seminal work. In particular, the links between the phase generation by reflections and the geometry of the characteristic variety will be the foundation of the proofs in this article (see section 3 and [17, section 6], for more details).

Indeed, in [17] the authors give examples of corner problems whose characteristic variety is such that, according to their argumentation, the associated ansatz of the geometric optics expansion has to contain more phases than the analogous ansatz for each problem in the half spaces $\{x_1 > 0\}$ and $\{x_2 > 0\}$. They also show that a new phenomenon, specific to corner problems, may happen for some characteristic variety configurations : the existence of "selfinteracting phases". By "selfinteracting phases" we mean that some phases can regenerate themselves after a suitable number of reflections on both sides of the corner. Such spectral configurations trap part of the solution in a periodically repeating pattern of reflections from one side to the other (see definition 4.5 and Figure 6 for more details).

Our aim is to give a *rigorous* construction of the geometric optics expansion when selfinteracting phases occur. This result is achieved in Theorem 4.2. The most interesting thing during this construction is the appearance of a new amplitude equation whose resolution is needed to initialize the resolution of the whole cascade of equations. More precisely, the resolution of the new amplitude equation requires the invertibility of an operator acting on the trace of one of the self-interacting amplitudes. This operator arises under the form $(I - \mathbb{T})$ and is reminiscent of Osher's invertibility assumption [14] for proving a priori estimate for (1). We show in Theorems 4.3 and 4.4 that a sufficient and necessary (in many meaningful cases) condition for the new amplitude equation to be solvable in $L^2(\mathbb{R}_+)$ is that the energy associated with the trapped information does not increase. Such a formulation matches with the naive (but intuitive) idea that if part of the information is trapped and increases after running through one cycle, then the associated geometric optics expansion will blow up after repeated cycles.

Inverting an operator of the form $(I - \mathbb{T})$ in view of constructing the geometric optics expansion is not surprising. Indeed, if we make the analogy with the analysis of the initial boundary value problem in the half space, the necessary and sufficient condition to ensure strong well-posedness is the so-called uniform Kreiss-Lopatinskii condition (see [7] and Assumption 2.3). When one wants to construct geometric optics expansions for such problems in a half space, a "microlocalized" version of this condition arises [19]. So one should expect that an analogous situation takes place for the corner problem and that the solvability condition we exhibit here is a "microlocalized" version of a stronger condition ensuring well-posedness of (1).

The full characterization of strong well-posedness for the corner problem has not been achieved yet. Some partial results are known, for example for symmetric corner problems with strictly dissipative boundary conditions (in that framework the strong well-posedness can easily be obtained with few modifications of the proofs of [10] and [3] for half space problems). However there are, to our knowledge, few results concerning the general framework, that is to say corner problems only satisfying the uniform Kreiss-Lopatinskii condition on each side. A fundamental contribution to this study is the article by Osher [14]. In this paper, the author uses the invertibility of an operator reading $(I - \mathbb{T}_\zeta)$-here ζ denotes a time frequency-to establish *a priori* energy estimates. More precisely, he uses such an invertibility property to construct a "Kreiss type symmetrizer" providing *a priori* energy estimates with a loss of regularity from the source terms to the solution. Unfortunately the number of losses in the estimate it not even explicit. However, some new results about the possibility to obtain energy estimates without loss can be found in [2].

We believe that, as for the half space problems, the invertibility condition on $(I - \mathbb{T})$ is a microlocalized version of Osher's condition. It is also interesting to remark that the example given in paragraph 3.5 shows that the invertibility condition on $(I - \mathbb{T})$ may not be satisfied if we only impose the uniform Kreiss-Lopatinskii condition on either side of the corner. But, looking still at the example of paragraph 3.5, we observe that the invertibility condition on $(I - \mathbb{T})$ is automatically satisfied if the boundary conditions are strictly dissipative.

The paper is organized as follows : in section 2 we define some objects and introduce notations for dealing with geometric optics expansions for initial boundary value problems. We also give in section 2 some known results about the well-posedness theory for the corner problem (1). In section 3, we explain, and make complete, the phase generation process by reflection as studied in [17]. We also briefly give an example of a 2×2 corner problem for which geometric optics expansions contain infinitely many phases.

Section 4 is devoted to the proof of our main result. Firstly, we give a rigorous framework for the description of the phases obtained by successive reflections. This framework has to be general enough to take into account selfinteracting phases. Then we construct the geometric optics expansion. To do that it is, in a first time, necessary to exhibit a global "tree" structure on the set of phases, then to find a way to initialize the resolution. As already mentionned, the initialization needs solving a new amplitude equation for the trace of a selfinteracting amplitude. The derivation of this equation is performed in paragraph 4.2.2. Then we show that, once we have organized the set of phases and we have constructed one of the selfinteracting amplitudes, we can construct all amplitudes associated with phases "close to" the selfinteracting ones. A more precise study of the structure of the phase set then permits to determine all the phases in the geometric optics expansion.

The end of section 4 aims at justifying the geometric optics expansion and then at giving a necessary and sufficient condition to ensure that the operator $(I - \mathbb{T})$ is invertible. We also give examples of corner problems with one loop and revisit some of the conclusions of [17]. Eventually, we make some comments on our results and give some prospects in setion 5.

2 An overview of well-posedness for half space and corner problems.

2.1 Notations and definitions.

Let

$$\Omega := \left\{(x_1, x_2) \in \mathbb{R}^2 \setminus x_1 \geq 0, x_2 \geq 0\right\}, \ \partial\Omega_1 := \Omega \cap \{x_1 = 0\}, \text{ and } \partial\Omega_2 := \Omega \cap \{x_2 = 0\},$$

be the quarter space and both its edges. For $T > 0$, we will denote :

$$\Omega_T :=]-\infty, T] \times \Omega, \ \partial\Omega_{1,T} :=]-\infty, T] \times \partial\Omega_1, \text{ and } \partial\Omega_{2,T} :=]-\infty, T] \times \partial\Omega_2.$$

The used function spaces the usual Sobolev spaces $H^n(X)$, with the notations $L^2(X) = H^0(X)$ and $H^\infty(X) := \cap_n H^n(X)$, where X is some Banach space. But we will also need the weighted Sobolev spaces defined by : for $\gamma > 0$, $H^n_\gamma(X) := \{u \in \mathcal{D}'(X) \setminus e^{-\gamma t}u \in H^n(X)\}$.

At last, during the construction of the WKB expansion, to make sure that amplitudes are smooth enough, we shall need the source term in (1) to be flat at the corner. The associated space of profiles is thus defined as : for $n \in \mathbb{N} \cup \{\infty\}$,

$$H_f^n := \left\{g \in H^n(\mathbb{R} \times \mathbb{R}_+) \setminus \forall k \geq n,\ \partial_x^k g(t,x)_{|x=0} = 0\right\}. \tag{2}$$

The flat at the corner weighted Sobolev spaces $H^n_{f,\gamma}$ are defined in a similar way.

Hereof $\mathscr{L}$ will be the symbol of the differential operator $L(\partial)$, *i.e.* for $\tau \in \mathbb{R}$ and $\xi \in \mathbb{R}^2$:

$$\mathscr{L}(\tau, \xi) := \tau I + \sum_{j=1}^{2} \xi_j A_j.$$

The characteristic variety V of $L(\partial)$ is given by :

$$V := \left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 \setminus \det \mathscr{L}(\tau, \xi) = 0\right\}.$$

In this article we choose to work with constantly hyperbolic operators. However it had to be mentioned that the analysis of section 4 is slightly easier in the particular framework of stricly hyperbolic operators. We thus assume the following property on $L(\partial)$:

Assumption 2.1 *There exists an integer $q \geq 1$, real valued $\lambda_1, ..., \lambda_q$ analytic on $\mathbb{R}^2 \setminus \{0\}$ and positive integers $\mu_1, ..., \mu_q$ such that :*

$$\forall \xi \in \mathbb{S}^1, \det \mathscr{L}(\tau, \xi) = \prod_{j=1}^{q} (\tau + \lambda_j(\xi))^{\mu_j},$$

where the semi-simple eigenvalues $\lambda_j(\xi)$ satisfy $\lambda_1(\xi) < ... < \lambda_q(\xi)$.

Let us also assume that the boundary of Ω is non-characteristic, and that the matrices B_1 and B_2 induce the good number of boundary conditions, that is to say :

Assumption 2.2 *We assume that the matrices A_1, A_2 are invertible. Then p_1 (resp. p_2), the number of lines of B_1, equals the number of positive eigenvalues of A_1 (resp. A_2).*

Moreover we also assume that B_1 and B_2 are of maximal rank.

Under Assumptions 2.1 and 2.2, we can define the resolvent matrices :

$$\mathscr{A}_1(\zeta) := -A_1^{-1}(\sigma I + i\eta A_2) \text{ and } \mathscr{A}_2(\zeta) := -A_2^{-1}(\sigma I + i\eta A_1),$$

where ζ denotes an element of the frequency space :

$$\Xi := \{\zeta := (\sigma = \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}, \gamma \geq 0\} \setminus \{(0,0)\}.$$

For convenience, we also introduce Ξ_0 the boundary of Ξ :

$$\Xi_0 := \Xi \cap \{\gamma = 0\}.$$

For $j = 1, 2$, $\zeta \in (\Xi \setminus \Xi_0)$, we denote by $E_j^s(\zeta)$ the stable subspace of $\mathscr{A}_j(\zeta)$ and $E_j^u(\zeta)$ its unstable subspace. These spaces are well-defined according to [6]. The stable subspace $E_j^s(\zeta)$ has dimension p_j, whereas $E_j^u(\zeta)$ has dimension $N - p_j$. Let us recall the following Theorem due to Kreiss [7] and generalized by Métivier [12] for constantly hyperbolic operators :

Theorem 2.1 (block structure) *Under Assumptions 2.1 and 2.2, for all $\underline{\zeta} \in \Xi$, there exists a neighborhood $\mathscr{V}$ of $\underline{\zeta}$ in Ξ, integers $L_1, L_2 \geq 1$, two partitions $N = \nu_{1,1} + ... + \nu_{1,L_1} = \nu_{2,1} + ... + \nu_{2,L_2}$ with $\nu_{1,l}, \nu_{2,l} \geq 1$, and two invertible matrices T_1, T_2, regular on $\mathscr{V}$ such that :*

$$\begin{aligned} \forall \zeta \in \mathscr{V},\quad T_1(\zeta)^{-1}\mathscr{A}_1(\zeta)T_1(\zeta) &= diag\left(\mathscr{A}_{1,1}(\zeta), ..., \mathscr{A}_{1,L_1}(\zeta)\right), \\ T_2(\zeta)^{-1}\mathscr{A}_2(\zeta)T_2(\zeta) &= diag\left(\mathscr{A}_{2,1}(\zeta), ..., \mathscr{A}_{2,L_2}(\zeta)\right), \end{aligned}$$

where the blocks $\mathscr{A}_{j,l}(\zeta)$ have size $\nu_{j,l}$ and satisfy one of the following alternatives :

i) All the elements in the spectrum of $\mathscr{A}_{j,l}(\underline{\zeta})$ have positive real part.

ii) All the elements in the spectrum of $\mathscr{A}_{j,l}(\underline{\zeta})$ have negative real part.

iii) $\nu_{j,l} = 1$, $\mathscr{A}_{j,l}(\underline{\zeta}) \in i\mathbb{R}$, $\partial_\gamma \mathscr{A}_{j,l}(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$, and $\mathscr{A}_{j,l}(\zeta) \in i\mathbb{R}$ for all $\zeta \in \mathscr{V} \cap \Xi_0$.

iv) $\nu_{j,l} > 1$, $\exists k_{j,l} \in i\mathbb{R}$ such that

$$\mathscr{A}_{j,l}(\underline{\zeta}) = \begin{bmatrix} k_{j,l} & i & 0 \\ & \ddots & i \\ 0 & & k_{j,l} \end{bmatrix},$$

the coefficient in the lower left corner of $\partial_\gamma \mathscr{A}_{j,l}(\underline{\zeta})$ is real and non-zero, and moreover $\mathscr{A}_{j,l}(\zeta) \in iM_{\nu_{j,l}}(\mathbb{R})$ for all $\zeta \in \mathscr{V} \cap \Xi_0$.

Thanks to this Theorem it is possible to describe the four kinds of frequencies, for each part of the boundary $\partial\Omega$:

Definition 2.1 *For $j = 1, 2$, we denote by :*

1) $\mathscr{E}_j$ the set of elliptic frequencies, that is to say the set of $\underline{\zeta} \in \Xi_0$ such that Theorem 2.1 for the matrix $\mathscr{A}_j(\underline{\zeta})$ is satisfied with one block of type i) and one block of type ii) only.

2) $\mathscr{H}_j$ the set of hyperbolic frequencies, that is to say the set of $\underline{\zeta} \in \Xi_0$ such that Theorem 2.1 for the matrix $\mathscr{A}_j(\underline{\zeta})$ is satisfied with blocks of type iii) only.

3) $\mathscr{E}\mathscr{H}_j$ the set of mixed frequencies, that is to say the set of $\underline{\zeta} \in \Xi_0$ such that Theorem 2.1 for the matrix $\mathscr{A}_j(\underline{\zeta})$ is satisfied with one block of type i), one of type ii) and at least one of type iii), but without block of type iv).

4) $\mathscr{G}_j$ the set of glancing frequencies, that is to say the set of $\underline{\zeta} \in \Xi_0$ such that Theorem 2.1 for the matrix $\mathscr{A}_j(\underline{\zeta})$ is satisfied with at least one block of type iv).

Thus, by definition, Ξ_0 admits the following decomposition :

$$\Xi_0 = \mathscr{E}_j \cup \mathscr{E}\mathscr{H}_j \cup \mathscr{H}_j \cup \mathscr{G}_j.$$

The study made in [7] and in [12] shows that the subspaces $E_1^s(\zeta)$ and $E_2^s(\zeta)$ admit a continous extension up to Ξ_0. Moreover, if $\underline{\zeta} \in \Xi_0 \setminus (\mathscr{G}_1 \cup \mathscr{G}_2)$ one can decompose :

$$\mathbb{C}^N = E_1^s(\underline{\zeta}) \oplus E_1^u(\underline{\zeta}) = E_2^s(\underline{\zeta}) \oplus E_2^u(\underline{\zeta}), \tag{3}$$

and for $j \in \{1, 2\}$:

$$E_j^s(\underline{\zeta}) = E_j^{s,e}(\underline{\zeta}) \oplus E_j^{s,h}(\underline{\zeta}),\ E_j^u(\underline{\zeta}) = E_j^{u,e}(\underline{\zeta}) \oplus E_j^{u,h}(\underline{\zeta}).$$

where $E_j^{s,e}(\underline{\zeta})$ (resp. $E_j^{u,e}(\underline{\zeta})$) is the generalized eigenspace associated with eigenvalues of $\mathscr{A}_j(\underline{\zeta})$ with negative (resp. positive) real part, and where the spaces $E_j^{s,h}(\underline{\zeta})$ and $E_j^{u,h}(\underline{\zeta})$ are sums of eigenspaces of $\mathscr{A}_j(\underline{\zeta})$ associated with some purely imaginary eigenvalues of $\mathscr{A}_j(\underline{\zeta})$. From assumption 2.2 we also have :

$$\mathbb{C}^N = A_1 E_1^s(\underline{\zeta}) \oplus A_1 E_1^u(\underline{\zeta}) = A_2 E_2^s(\underline{\zeta}) \oplus A_2 E_2^u(\underline{\zeta}). \tag{4}$$

In fact, it is possible to give a more precise decomposition of the spaces $E_j^{s,h}(\underline{\zeta})$ and $E_j^{u,h}(\underline{\zeta})$. Indeed, let $\underline{\omega}_{m,j}$ be a purely imaginary eigenvalue of $\mathscr{A}_j(\underline{\zeta})$, that is :

$$\det(\underline{\tau} + \underline{\eta} A_1 + \underline{\omega}_{m,2} A_2) = \det(\underline{\tau} + \underline{\omega}_{m,1} A_1 + \underline{\eta} A_2) = 0.$$

Then, using Assumption 2.1, there exists an index $k_{m,j}$ such that :

$$\underline{\tau} + \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) = \underline{\tau} + \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) = 0,$$

where $\lambda_{k_{m,j}}$ is smooth in both variables. Let us then introduce the following classification :

Definition 2.2 *The set of incoming (resp. outgoing) phases for the side $\partial\Omega_1$, denoted by $\mathfrak{I}_1$ (resp. $\mathfrak{O}_1$), is the set of indeces m such that the group velocity $v_m := \nabla\lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta})$ satisfies $\partial_1\lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) > 0$ (resp. $\partial_1\lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) < 0$).*

Similarly, the set of incoming (resp. outgoing) phases for the side $\partial\Omega_2$, denoted by $\mathfrak{I}_2$ (resp. $\mathfrak{O}_2$, is the set of indeces m such that the group velocity $v_m := \nabla\lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2})$ satisfies $\partial_2\lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) > 0$ (resp. $\partial_2\lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) < 0$).

Thanks to this definition, we can write the following decomposition of the stable and unstable components $E_j^{s,h}(\underline{\zeta})$ and $E_j^{u,h}(\underline{\zeta})$:

Lemma 2.1 *For all $\underline{\zeta} \in \mathscr{H}_j \cup \mathscr{E}\mathscr{H}_j$, $j = 1, 2$ there holds*

$$E_1^{s,h}(\underline{\zeta}) = \oplus_{m \in \mathfrak{I}_1} \ker \mathscr{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \quad E_1^{u,h}(\underline{\zeta}) = \oplus_{m \in \mathfrak{O}_1} \ker \mathscr{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \tag{5}$$

$$E_2^{s,h}(\underline{\zeta}) = \oplus_{m \in \mathfrak{I}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}), \quad E_2^{u,h}(\underline{\zeta}) = \oplus_{m \in \mathfrak{O}_2} \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}). \tag{6}$$

From Assumption 2.2 we can also write :

$$A_1 E_1^{s,h}(\underline{\zeta}) = \oplus_{m \in \mathfrak{I}_1} A_1 \ker \mathscr{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \quad A_1 E_1^{u,h}(\underline{\zeta}) = \oplus_{m \in \mathfrak{O}_1} A_1 \ker \mathscr{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \tag{7}$$

$$A_2 E_2^{s,h}(\underline{\zeta}) = \oplus_{m \in \mathfrak{I}_2} A_2 \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}), \quad A_2 E_2^{u,h}(\underline{\zeta}) = \oplus_{m \in \mathfrak{O}_2} A_2 \ker \mathscr{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}). \tag{8}$$

We refer, for example, to [5] for a proof of this lemma.

2.2 Known results about strong well-posedness.

We consider the corner problem with source terms in the interior of Ω_T and on either side of the boundary $\partial\Omega_T$, it reads :

$$\begin{cases} L(\partial)u = f, \text{ on } \Omega_T, \\ B_1 u_{|x_1=0} = g_1, \text{ on } \partial\Omega_{1,T}, \\ B_2 u_{|x_2=0} = g_2, \text{ on } \partial\Omega_{2,T}, \\ u_{|t\leq 0} = 0. \end{cases} \tag{9}$$

By strong well-posedness for the corner problem (9) we mean the following :

Definition 2.3 *The corner problem* (9) *is said to be strongly well-posed if for $T > 0$, for all $f \in L^2(\Omega_T)$,$g_j \in L^2(\partial\Omega_{j,T})$, the corner problem* (9) *admits a unique solution $u \in L^2(\Omega_T)$ with traces in $L^2(\partial\Omega_{1,T})$ and $L^2(\Omega_{2,T})$ satisfying the following energy estimate :*

$$\|u\|^2_{L^2(\Omega_T)} + \|u_{|x_1=0}\|^2_{L^2(\partial\Omega_{1,T})} + \|u_{|x_2=0}\|^2_{L^2(\partial\Omega_{2,T})} \leq C_T \left(\|f\|^2_{L^2(\Omega_T)} + \|g_1\|^2_{L^2(\partial\Omega_{1,T})} + \|g_2\|^2_{L^2(\partial\Omega_{2,T})} \right), \tag{10}$$

for some constant C_T depending on T.

As we have already mentionned in the introduction, the full characterization of strong well-posedness for the corner problem (9) has not been achieved yet. However we have some partial results.

First of all, the strong well-posedness is proved in the particular framework of symmetric operators with strictly dissipative boundary conditions, that is boundary conditions defined as follows :

Definition 2.4 *For $j = 1, 2$, the boundary condition $B_j u_{|x_j=0} = g_j$ is said to be strictly dissipative if the following inequality holds :*

$$\forall v \in \ker B_j \setminus \{0\}, \ \langle A_j v, v \rangle < 0$$

and $\ker B_j$ is maximal (in the sense of inclusion) for this property.

We thus have the following result :

Theorem 2.2 *[2, chapter 4]Under Assumption 2.2, if the matrices A_1 and A_2 are symmetric and if the boundary conditions of the corner problem* (9) *are strictly dissipative, then under a certain algebraic condition on the matrix $A_1^{-1}A_2$, the corner problem* (9) *is strongly well-posed is the sense of Definition 2.3.*

We refer to [2, chapter 4] for a proof of this result and for more details about the mentionned algebraic condition (see Assumption 4.1.2 of [2])[1].

It is also easy to show (see [2, paragraph 5.3.1]) that a necessary condition for (9) to be strongly well-posed is that each initial boundary value problem :

$$\begin{cases} L(\partial)u = f, \text{ on } \{x_j > 0, x_{3-j} \in \mathbb{R}\}, \\ B_j u_{|x_j=0} = g_j, \\ u_{|t\leq 0} = 0, \end{cases} \tag{11}$$

is strongly well-posed in the usual sense for initial boundary value problems in the half-space (see for example [3]).

This implies that Theorem 2.2 is not sharp (except for $N = 2$ thanks to [18]) because there exist non-stricly dissipative boundary conditions leading to a strongly well-posed initial boundary value problem (11) (see for example [1, paragraph 5.3]).

However, the set of the boundary conditions making (11) strongly well-posed has been characterized by [7] and is composed of the boundary condition satisfying the so-called uniform Kreiss-Lopatinskii condition :

Definition 2.5 *The initial boundary value problem* (11) *is said to satisfy the uniform Kreiss-Lopatinskii condition if for all* $\zeta \in \Xi$*, we have*

$$\ker B_j \cap E_j^s(\zeta) = \{0\}\,.$$

So for the corner problem (9) to be strongly well-posed it is necessary that for $j = 1, 2$, the initial boundary value problem (11) satisfies the "uniform" Kreiss-Lopatinskii condition. We thus make the assumption :

Assumption 2.3 *For all* $\zeta \in \Xi$*, we have*

$$\ker B_1 \cap E_1^s(\zeta) = \ker B_2 \cap E_2^s(\zeta) = \{0\}\,.$$

In particular, the restriction of B_1 *(resp.* B_2*) to the stable subspace* $E_1^s(\zeta)$ *(resp.* $E_2^s(\zeta)$*) is invertible, its inverse is denoted by* $\phi_1(\zeta)$ *(resp.* $\phi_2(\zeta)$*).*

Unsurprisingly, the cunterexample [15] shows that imposing the uniform Kreiss-Lopatinskii condition on each side of the boundary is not sufficient to ensure that the corner problem (9) is strongly well-posed.

3 The phase generation process and examples.

Before constructing the geometric optics expansions, it is necessary to describe the expected phases in these expansions. Since the boundary of the domain Ω is not flat, we expect that it is possible to generate more phases than for half space problems. Indeed, at the very first glance, we can think that a ray of geometric optics can be reflected several times on the boundary of the domain, with different new phases generated at each reflection.

It is thus very important in order to postulate an ansatz to be able to describe all the phases that can be obtained by successive reflections on each side of the boundary.

Here, we shall go back to the discussion by Sarason and Smoller in [17] explaining this phenomenon and establishing a very strong link between the geometry of the characteristic variety of $L(\partial)$ and the phase generation process.

As already mentionned in the introduction, we are interested here in corner problems which are homogeneous in the interior and on one side of the boundary. The only non-zero source term, which arises in the boundary condition on $\partial\Omega_1$, will be highly oscillating, and we want to understand which phases can be induced by this source term. We will here describe the phase generation process when the source term is taken on $\partial\Omega_1$; the arguments are the same for a source term on $\partial\Omega_2$.

[1] We do not want to give more details about this condition because it is not used to construct the WKB expansion. Moreover, this condition will be satisfy by all our examples.

3.1 Source term induced phases.

Our problem of study reads :

$$\begin{cases} L(\partial)u^\varepsilon = 0, \text{ on } \Omega_T, \\ B_1 u^\varepsilon_{|x_1=0} = g^\varepsilon, \text{ on } \partial\Omega_{1,T} \\ B_2 u^\varepsilon_{|x_2=0} = 0, \text{ on } \partial\Omega_{2,T} \\ u^\varepsilon_{|t\leq 0} = 0, \end{cases} \tag{12}$$

where the source term on $\partial\Omega_{1,T}$ is given by :

$$g^\varepsilon(t,x_2) := e^{\frac{i}{\varepsilon}\varphi(t,x_2)} g(t,x_2), \tag{13}$$

where the amplitude $g \in H^\infty_f$, and is zero for negative times. The planar phase φ is defined by :

$$\varphi(t,x_2) := \underline{\tau} t + \underline{\xi}_2 x_2,$$

for two fixed real numbers $\underline{\tau} > 0$ and $\underline{\xi}_2$.

The fact that g belongs to H^∞_f implies that g^ε is zero at the corner. Assume that g identically vanishes in a neighborhood of the corner. Then by finite speed of propagation for the half-space problem, we can, at least during a small time interval, see the corner problem (12) as a boundary value problem in the half space $\{x_1 \geq 0\}$.

Geometric optics expansions for boundary value problem in the half space have already been studied (see for example [19]) and, going back over the existing analysis, we expect that the source term g^ε on the side $\partial\Omega_1$ induces in the interior of the domain several rays associated with the planar phases :

$$\varphi^{0,k}(t,x) := \varphi(t,x_2) + \underline{\xi}_1^{0,k} x_1,$$

where the $(\underline{\xi}_1^{0,k})_k$ are the roots in the ξ_1 variable to the dispersion relation :

$$\det \mathscr{L}(\underline{\tau}, \xi_1, \underline{\xi}_2) = 0. \tag{14}$$

An important remark to understand the phase generation process is that the $(\underline{\xi}_1^{0,k})_k$ are the intersection points (with the convention that complex roots are viewed as intersection points at infinity) between the line $\left\{(\underline{\tau}, \xi_1, \underline{\xi}_2), \xi_1 \in \mathbb{R}\right\}$ and the section of the characteristic variety V at $\tau = \underline{\tau}$.

Let us denote by p_r the number of real roots to (14) and by $2p_c$ the number of complex roots (which occur in conjugate pairs). We also assume that $(\underline{\tau}, \underline{\xi}_2)$ is not a glancing frequency for the matrix $\mathscr{A}_1$, hence p_r can be decomposed as $p_r^i + p_r^o$, where p_r^i (resp. p_r^o) is the number of real roots inducing an incoming (resp. outgoing) group velocity (see definition 2.2). We thus have $p_1 = p_r^i + p_c$ and $N - p_1 = p_r^o + p_c$ by Theorem 2.1. Firstly we shall consider φ_i^0, one of the p_r^i phases with an incoming group velocity and φ_o^0, one of the p_r^o phases with an outgoing group velocity. We also denote by v_i^0 and v_o^0 the associated group velocity. Phases associated with complex roots will be dealt with in a second time. The following discussion should be performed for each such real phase.

We shall study separately the influence of the phases φ_i^0 and φ_o^0 upon the generation of phases.

$\diamond$ The phase φ_o^0.

The phase φ_o^0, associated with an outgoing group velocity, describes the "past" of the information reflected on the side $\partial\Omega_1$ at the initial time. In other words, to know the origin of a point on the side $\partial\Omega_1$, it is sufficient to travel along the characteristic with group velocity v_o^0 by rewinding time back to $-\infty$.

This leads us to separate two cases, making, thus, more precise the definition 2.2 :

Definition 3.1 *An outgoing group velocity $v = (v_1, v_2)$ for the side $\partial\Omega_1$ (i.e. $v_1 < 0$) is said to be :*

- *outgoing-incoming if $v_2 > 0$.*
- *outgoing-outgoing if $v_2 < 0$.*

• First subcase : v_o^0 outgoing-outgoing.
Let us fix a point on the side $\partial\Omega_1$ and we draw the characteristic line with group velocity v_o^0 passing through this point. Since v_o^0 is outgoing for each side of the boundary, the information at the considered point of $\partial\Omega_1$ can only come from information in the interior of the domain which has been transported towards the side $\partial\Omega_1$ see Figure 1. But, without source term in the interior of Ω such information is zero. As a consequence, the amplitude u_o^0 associated with the phase φ_o^0 is zero, since according to Lax' lemma [8] it satisfies the transport equation :

$$\begin{cases} \partial_t u_o^0 + v_o^0 \cdot \nabla_x u_o^0 = 0, \\ u^0_{o_{|t\leq 0}} = 0. \end{cases}$$

Outgoing-outgoing phases do not have any influence on the WKB expansion or on the phase generation process and are therefore ignored from now on.

• Second subcase : v_o^0 outgoing-incoming.
Once again, we fix a point on the side $\partial\Omega_1$ and we draw the characteristic line with group velocity v_o^0 passing through this point. As in the subcase of an outgoing-outgoing, the information at the considered point of $\partial\Omega_1$ can not come from the interior of the domain.

However, the characteristic associated with the group velocity v_o^0 hits the side $\partial\Omega_2$ when we rewind the time back to $-\infty$, so the information at the point of the side $\partial\Omega_1$ could *a priori* come from some information on the side $\partial\Omega_2$ which would have been transported towards the side $\partial\Omega_1$. But this is not possible at time $t = 0$ since the boundary condition on $\partial\Omega_2$ is homogeneous for negative times.

So, the amplitude associated with the outgoing-incoming phase φ_o^0 is zero at time $t = 0$ and even on a small time interval if g^ε identically vanishes near the corner. That is why we do not take into account the phase φ_o^0 **initially** in the phase generation process.

Let us stress here that the phase φ_o^0 is moved apart *a priori* only at time $t = 0$. Indeed, for some configuration of the characteristic variety, this phase can be generated at a future reflection on the side $\partial\Omega_2$, and will finally be included in the ansatz. We will make more comments on this point at paragraph 3.3, after having precisely described which reflections are taken into account.

$\diamond$ The phase φ_i^0.

The phase φ_i^0 is associated with an incoming group velocity for the side $\partial\Omega_1$. Opposite to the phase φ_o^0, it describes the "future" of the source term g^ε. That is to say, when time goes to $+\infty$, part of the oscillations in g^ε is transported along the characteristic with group velocity v_i^0. So, the phase φ_i^0 carries a non-zero information and has to be taken into account in the phase generation process.

However, once again, we have to separate two subcases, according to the following refinement of the definition 2.2 :

Definition 3.2 *An incoming group velocity $v = (v_1, v_2)$ for the side $\partial\Omega_1$ (i.e. $v_1 > 0$) is said to be :*
• *incoming-incoming if $v_2 > 0$.*
• *incoming-outgoing if $v_2 < 0$.*

The four kinds of (non-glancing) oscillating phases used in this analysis are drawn in Figure 1.

• First subcase : v_i^0 incoming-incoming.

We choose a point $(0, \underline{x}_2)$, on $\partial\Omega_1$ such that $g^\varepsilon(0, \underline{x}_2)$ is non-zero and we draw the characteristic with velocity v_i^0 passing through this point. When t goes to $+\infty$, the information transported along this ray will never hit the side $\partial\Omega_2$ and will be unable to generate new phases by reflection. So, when the group velocity v_i^0 is incoming-incoming, the phase generation process for the phase φ_i^0 stops.

• Second subcase : v_i^0 incoming-outgoing.

We fix a point $(0, \underline{x}_2) \in \partial\Omega_1$ with $g^\varepsilon(0, \underline{x}_2) \neq 0$, and we draw the characteristic with velocity v_i^0 passing through this point. As $v_{i,2}^0$ is negative, this ray will hit, after a while, the side $\partial\Omega_2$. We thus expect that this ray will give rise to reflected oscillations and that this reflection will create new phases. This reflection phenomenon and more specifically the new expected phases will be described in the next paragraph. But before that, we will conclude the discussion on the phases induced by the source term g^ε by considering the possible complex valued phases.

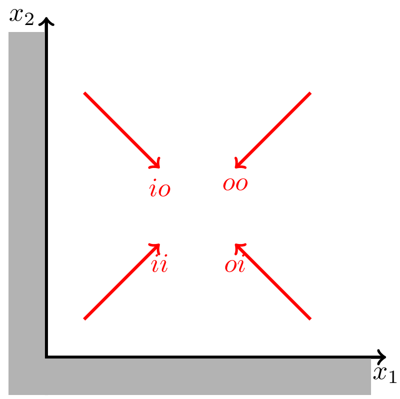

Figure 1: The four different kinds of phases.

$\diamond$ φ_k^0 complex valued.

We now deal with phases corresponding to roots in the ξ_1 variable to the dispersion relation (14) with non-zero imaginary part. Let us first introduce some vocabulary.

Definition 3.3 *A phase φ_k^0 with $\underline{\xi}_1^{0,k} \in \mathbb{C} \setminus \mathbb{R}$ is said to be :*

- *evanescent for the side $\partial\Omega_1$ if Im $\underline{\xi}_1^{0,k} > 0$.*
- *explosive for the side $\partial\Omega_1$ if Im $\underline{\xi}_1^{0,k} < 0$.*

Thanks to the construction of geometric optics expansion for complex valued phases made for example in [11] and [9], the expected behaviour of the amplitudes associated with these phases is a propagation of information in the normal direction to the side $\partial\Omega_1$. However, this propagation is exponentially decreasing (resp. increasing) according to the variable x_1 for the amplitudes linked with evanescent (resp. explosive). In all that follows, as we are looking for amplitudes in $L^2(\Omega)$ so as in [9],[11] and [19] we do not take into account explosive phases.

Thus, we only keep the evanescent phases. Since, for regularity considerations on the oscillating amplitudes, we are working with a source term in H_f^∞, this source term satisfies, in particular, $g(t, 0) = 0$. Consequently, the information carried by evanescent phases will never hit the side $\partial\Omega_2$ and the evanescent phases for the side $\partial\Omega_1$ are, as well as the incoming-incoming ones, stopping conditions in the phase generation process.

To summarize, the phases induced directly by the source term g^ε are the incoming (for the side $\partial\Omega_1$) phases and the evanescent phases for the side $\partial\Omega_1$. Incoming-incoming and evanescent phases will not be reflected, thus we only have to study the reflections on $\partial\Omega_2$ associated with incoming-outgoing phases.

3.2 The first reflection.

We assume that the dispersion relation (14) has at least one solution in the ξ_1 variable generating an incoming-outgoing group velocity. We shall describe the reflection of one of these phases. Of course, to determine all the expected phases in the WKB expansion, the following discussion has to be repeated for each of these phases.

Let $\underline{\xi}_1^0$ be a fixed root in the ξ_1 variable to (14). We denote by v_i^0 the associated incoming-outgoing group velocity which corresponds to rays emanating from $\partial\Omega_1$ and hitting $\partial\Omega_2$ in finite time. Let us also assume that time is large enough so that the ray associated with v_i^0 and emanating from the support of g^ε has hit $\partial\Omega_2$. Once again, by (formal) finite speed of propagation arguments, the reflection of the ray can not hit immediatly the side $\partial\Omega_1$. Thus during a small time, we can represent our situation as an initial boundary value problem in the half space $\{x_2 \geq 0\}$ whose boundary source term has been turned on by the amplitude for the outgoing (for the side $\partial\Omega_2$) phase φ_i^0.

We thus have to determine the roots $(\underline{\xi}_2^{1,k})_k$ in the ξ_2 variable to the dispersion relation :

$$\det \mathscr{L}(\underline{\tau}, \underline{\xi}_1^0, \xi_2) = 0. \tag{15}$$

Let us stress that we already know one of them, that is $\underline{\xi}_2$. For each of the new roots, we associate the phase :

$$\varphi^{1,k}(t,x) := \underline{\tau} t + \underline{\xi}_1^0 x_1 + \underline{\xi}_2^{1,k} x_2.$$

It is interesting to note that the $(\underline{\xi}_2^{1,k})_k$ are the intersection points between $V \cap \{\tau = \underline{\tau}\}$ and the line $\left\{(\underline{\tau}, \underline{\xi}_1^0, \xi_2), \xi_2 \in \mathbb{R}\right\}$. Thus to determine the phases generated by the source term, it has been necessary to consider the intersection of $V \cap \{\tau = \underline{\tau}\}$ with a horizontal line, and to determine the phases generated by the first reflection we have to consider the intersection of $V \cap \{\tau = \underline{\tau}\}$ with a vertical line (see Figure 2). To determine the phases generated by the second reflection, we will have to consider the intersection with a horizontal line and so on. We see that this process strongly depends on the geometry of the characteristic variety V.

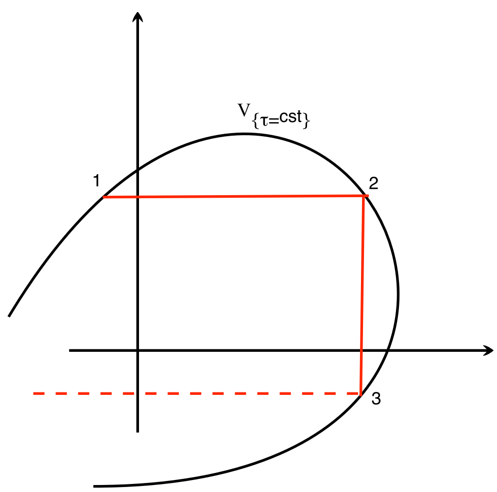

Figure 2: The geometry of the characteristic variety and the phase generation.

Repeating exactly the same arguments as those used for the phases induced by the source term, we claim that outgoing-outgoing and incoming-outgoing phases can be neglected (at least initially for incoming-outgoing phases). Consequently, for real roots to (15), we just have to consider those associated with an incoming-incoming or outgoing-incoming group velocity. Let φ_i^1 denote one of these phases and v_i^1 its group velocity.

$\diamond$ v_i^1 incoming-incoming.

In that case, as when the group velocity v_i^0 was incoming-incoming, the considered ray will never hit the side $\partial\Omega_1$, and it will never be reflected. The phase generation process for the phase φ_i^1 stops, and we are free to study the reflection(s) of another root to (15).

$\diamond$ v_i^1 outgoing-incoming.

The reflected ray travels towards $\partial\Omega_1$, it will hit $\partial\Omega_1$ after a while, and we will have to determine how it is reflected back. So the phase generation process for the phase φ_i^1 continues.

Concerning complex roots to (15) (if such roots exist), we only add in the WKB expansion those associated with evanescent phases for the side $\partial\Omega_2$ (that is to say those satisfying $\mathrm{Im}\ \underline{\xi}_2^{1,k} > 0$). As for the complex valued phases induced by the source term, they will never be reflected back and the phase generation process for these phases stops.

3.3 Summary.

To summarize the phase generation process is the following : we start from a source term on $\partial\Omega_1$ and we only study the reflections for the incoming phases that it induces. If all of the phases are incoming-incoming (or evanescent), then the process stops. Otherwise we determine the reflections on $\partial\Omega_2$ of all the incoming-outgoing phases and we shall consider them into the ansatz. If one of these reflected phases is outgoing-incoming we will determine its reflection on $\partial\Omega_1$ otherwise the phase generation process stops... This leads us to consider sequences of phases which are alternatively incoming-outgoing and outgoing-incoming until we find an incoming-incoming or evanescent phase during a reflection which ends the sequence.

There are, of course, two possibilities : either each of these sequences of phases generated by successive reflections is finite. Then the number of phases in the ansatz will be finite (see example of paragraph 3.5). Or at least one of these sequences is infinite, then the number of phases in the ansatz is infinite (see example 3.4).

In all the preceding discussion, we used the tacit assumption that we never meet glancing phases. This assumption is satisfied in all our examples and it will be clearly stated in Theorem 4.2. Formally glancing phases should be stopping criterion as well as incoming-incoming and evanescent phases. However, how to include rigorously glancing modes in the WKB expansion is left for future studies.

Let us also stress that during a reflection on the side $\partial\Omega_1$ (resp. $\partial\Omega_2$), the fact that outgoing-incoming (resp. incoming-outgoing) phases are not considered do not prevent these phases to appear in the WKB expansion.

Indeed, let $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$ be an incoming-outgoing phase generated by the source term g^ε and $\left(\underline{\tau}, \underline{\xi}_1, \widetilde{\underline{\xi}}_2\right)$ be an outgoing-incoming phase also generated by the source term. This phase is *a priori* not taken into account in the WKB expansion at the first step of the phase generation process described above. Let us assume that the intersection between the characteristic variety $V_{\tau=\underline{\tau}}$ and the line $\left\{(\underline{\tau}, \underline{\xi}_1, \xi_2), \xi_2 \in \mathbb{R}\right\}$ contains a value of ξ_2, $\widetilde{\underline{\xi}}_2$, such that the associated oscillating phase is outgoing-incoming and that the intersection between $V_{\tau=\underline{\tau}}$ and the line $\left\{(\underline{\tau}, \xi_1, \widetilde{\underline{\xi}}_2), \xi_1 \in \mathbb{R}\right\}$ contains the frequency $\left(\underline{\tau}, \widetilde{\underline{\xi}}_1, \widetilde{\underline{\xi}}_2\right)$ (in other words, it is equivalent to say that there exists a rectangle with sides parallel to the x and y axis whose corners are four points of $V_{\tau=\underline{\tau}}$). If the frequency $\left(\underline{\tau}, \widetilde{\underline{\xi}}_1, \widetilde{\underline{\xi}}_2\right)$ is associated with an incoming-outgoing group velocity, we remark by applying the phase generation process (more precisely during the third reflection), that we have to consider the frequency $\left(\underline{\tau}, \underline{\xi}_1, \widetilde{\underline{\xi}}_2\right)$ which has been initially excluded.

Moreover, when we study the reflections of the phase associated with the frequency $\left(\underline{\tau}, \widetilde{\underline{\xi}}_1, \widetilde{\underline{\xi}}_2\right)$ on the side $\partial\Omega_1$, we are led to consider one more time the phase with frequency $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$. So, the phase associated with the frequency $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$ is "selfgenerating" or "selfinteracting" because it is in the set of the phases that it generates. Such a configuration in the characteristic variety will be called a "loop". An explicit example of a corner problem with a loop will be given in paragraph 3.5.

The fact that at each reflection there is more than one generated phase and this selfinteraction phenomeon between the phases imply that there is no natural order on the set of phases as in the $N = 2$ framework. Indeed, when $N > 2$ we have to deal with a tree matching the phase generation at each reflection. Thus, constructing the WKB expansion when $N > 2$ will be less intuitive as when $N = 2$, a framework in which it is sufficient to use the order induced by the phase generation process. In paragraphs 4.2.1 and 4.2.4, we show how to overcome this lack of natural order in view of constructing the WKB expansion.

3.4 An example with infinitely many phases.

The aim of this paragraph is to illustrate the phase generation process and to give an explicit example of a corner problem whose geometric optics expansion contains an infinite number of phases. Moreover, this example will also stress the fact that the phase generation process is much simpler when $N = 2$, since it gives a natural order of construction of the WKB expansion.

Let us consider the corner problem (12) with

$$A_1 := \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix},\ A_2 := \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

It is thus clear that $p_1 = p_2 = 1$ then we have to choose $B_1, B_2 \in \mathbb{M}_{1,2}(\mathbb{R})$ in such a way that the boundary conditions on $\partial\Omega_1$ and $\partial\Omega_2$ are strictly dissipative [18]. Moreover one can easily check that this corner problem satisfies Assumptions 2.1, 2.2.

We choose for source term on $\partial\Omega_1$ in the corner problem (12) :

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}(t+\frac{1}{2}x_2)} g(t, x).$$

Then the phase generation process for this problem is precisely described in [2, paragraph 6.6.1], and is illustrated in Figure 3. The phases that we have to consider form a "stairway" in a parabola (see Figure

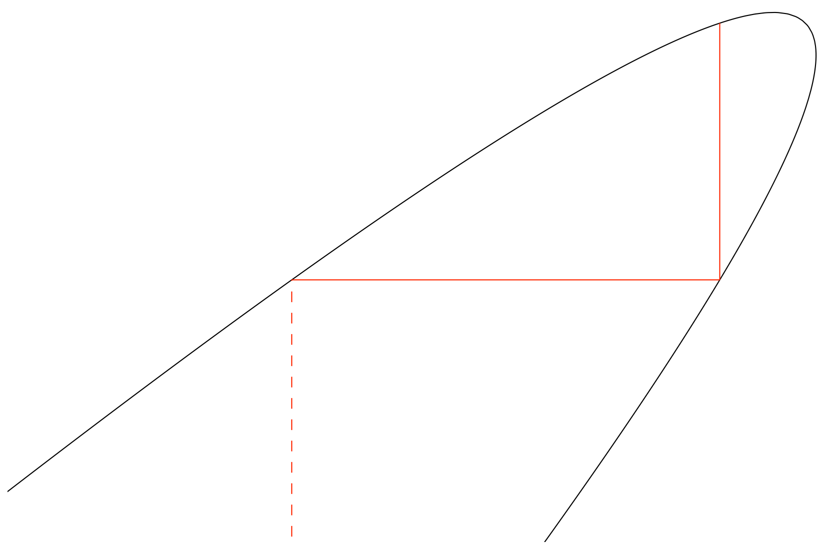

Figure 3: Phase generation for the corner problem of paragraph 3.4.

3). The points of this "stairway" are labelled by two sequences $(\xi_{1,p})_{p\in\mathbb{N}}$ and $(\xi_{2,p})_{p\in\mathbb{N}}$ in such a way that points $(\xi_{1,p}, \xi_{2,p})_{p\in\mathbb{N}}$ match with points in the "top of the parabola" whereas points $(\xi_{1,p}, \xi_{2,p+1})_{p\in\mathbb{N}}$ match with points in the "bottom of the parabola". Finally we initialize at $\xi_{1,0} = -\frac{1}{2}$ and $\xi_{2,0} = \frac{1}{2}$. A simple computation shows that we have :

$$\xi_{1,p} = -2p^2 - 3p - \frac{1}{2},\quad \xi_{2,p} = -2p^2 - p + \frac{1}{2},$$

and

$$\begin{aligned} v_p &= \frac{1}{4p^2 + 4p + 2} \begin{bmatrix} 4p+1 \\ -(4p+3) \end{bmatrix}, \\ w_p &= \frac{1}{4p^2 + 8p + 5} \begin{bmatrix} -(4p+5) \\ 4p+3 \end{bmatrix}. \end{aligned}$$

So all the points of the "top" are associated with incoming-outgoing group velocities while points of the "bottom" are associated with outgoing-incoming group velocities. Thus according to the phase generation process described in above the number of phases in the expansion will be infinite.

We refer to [2, paragraphs 6.6.2 and 6.6.3] for a rigorous construction of the geometric optics expansion and a justification of its convergence towards the exact solution. The difficult part of this analysis does not come from the construction because we are in the confortable case $N = 2$ but it comes from the justification. Indeed, when infinitely many phases occur, to ensure that the WKB expansion (at a finite order in terms of powers of ε) makes sense, we have to ensure that a series converges.

Eventually, let us remark the following phenomenon. If we fix a point $(0, y_p) \in \partial\Omega_1$ and follow the characteristic with group velocity v_p, then we will hit $\partial\Omega_2$ at time t_p^v in a point $(x_p, 0)$. Then if we start from $(x_p, 0) \in \partial\Omega_2$ and follow the characteristic line with group velocity w_p, will hit $\partial\Omega_1$ at time t_p^w in a point $(y_{p+1}, 0)$. A simple computation shows that the considered sequences are given by :

$$y_p = \frac{1}{4p+1} y_0, \;\; x_p = \frac{1}{4p+3} y_0, \;\; t_p^v = \frac{1}{v_{p,1}} y_p, \;\; t_p^w = \frac{1}{v_{p,2}} x_p,$$

from which we deduce that from the starting point $(0, y_0)$, $y_0 > 0$, we will get closer and closer of the corner at each reflection and will reach the corner in an infinite time. A scheme illustrating the characteristic lines for this corner problem is given in Figure 4.

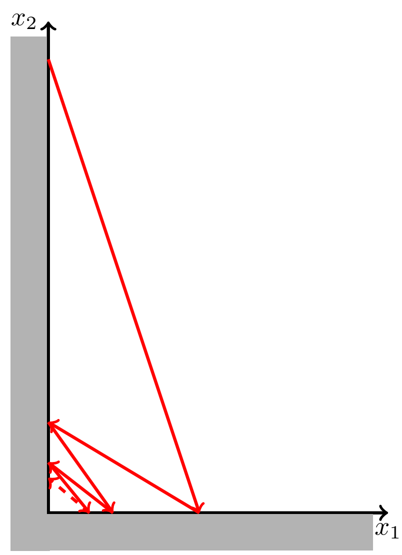

Figure 4: Appearance of the characteristics for problem of paragraph 3.4.

3.5 An example with a loop.

We consider the following corner problem :

$$\begin{cases} \partial_t u^\varepsilon + A_1 \partial_1 u^\varepsilon + A_2 \partial_2 u^\varepsilon = 0, \quad (x_1, x_2) \in \Omega, \\ B_1 u^\varepsilon_{|x_1=0} = 0, \\ B_2 u^\varepsilon_{|x_2=0} = g^\varepsilon, \\ u^\varepsilon_{|t\leq 0} = 0, \end{cases} \tag{16}$$

with :

$$A_1 := \begin{bmatrix} 0 & 0 & \sqrt{5} \\ 0 & \frac{5}{7} & 0 \\ \sqrt{5} & 0 & 4 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

This system does not have any physical meaning and is composed of a "wave type" equation and a scalar transport equation. It is clear that the corner problem (16) satisfies Assumption 2.2 with $p_1 = 2$ and $p_2 = 1$. The corner problem (16) does not satisfy Assumption 2.1, but it is hyperbolic in the sense of geometrically regular hyperbolic systems (see [13, definition 2.2]). This hyperbolicity assumption is sufficient for our discussion as long as we do not have to consider in the ansatz frequencies corresponding to intersection points[2] of the different sheets of the characteristic variety.

For the corner problem (16), the equation of the section of the characteristic variety with the plane $\{\tau = 1\}$ is given by :

$$(V_{\tau=1}) \quad \left(-5\xi_1^2 - 4\xi_2^2 + 4\xi_1\xi_2 + 4\xi_1 + 1\right)\left(1 + \frac{5}{7}\xi_1 - \xi_2\right) = 0,$$

and is composed of an ellipse and a crossing line, see Figure 5. If we choose for source term g^ε on $\partial\Omega_2$ in

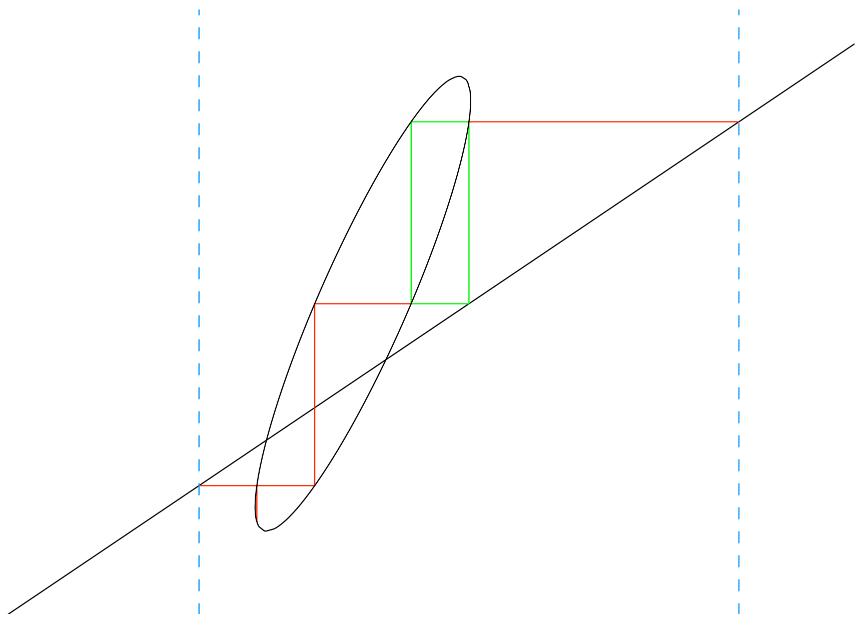

Figure 5: Section of the characteristic variety and the phase generation for corner problem (16)

(16) :

$$g^\varepsilon(t, x_1) = e^{\frac{i}{\varepsilon}\left(t + \frac{21}{5}x_1\right)} g(t, x_1).$$

then after application of the phase generation process (see Figure 5), we obtain a loop as introduced in

[2] We have to stress that these intersection points, specific to geometrically regular hyperbolic systems, can, generically, induce an infinite number of phases in the WKB expansion. Indeed, let us assume that in a given intersection point, one of the sheets of the variety is associated with an incoming-outgoing group velocity whereas the other sheet is associated with an outgoing-incoming group velocity. Then, using the fact that the group velocities are regular, one can find a neighborhood on each sheet such that the group velocity does not change type on this neighborhood. It immediatly follows that if a ray of the geometric optics expansion contains a frequency in these neighborhoods it is automatically attracted toward the intersection point by forming a "stairway" like in paragraph 3.4. The fact that this phenomenon does not occur for the corner problem (16), and that the number of generated phases in finite, is somewhat very special.

paragraph 3.3. The four selfinteracting phases and the associated group velocities are given by :

$$\begin{aligned} \varphi_1(t,x) &:= t+\frac{21}{5}x_1+8x_2,\ v_1=\frac{1}{\sqrt{5}}\begin{bmatrix}-3\\ \frac{2}{5}\end{bmatrix}, \\ \varphi_2(t,x) &:= t+\frac{21}{5}x_1+4x_2,\ v_2=\begin{bmatrix}\frac{5}{7}\\ -1\end{bmatrix}, \\ \varphi_3(t,x) &:= t+3x_1+4x_2,\ v_3=\begin{bmatrix}-5\\ 2\end{bmatrix}, \\ \varphi_4(t,x) &:= t+3x_1+8x_2,\ v_4=\begin{bmatrix}3\\ -2\end{bmatrix}, \end{aligned} \tag{17}$$

v_1 and v_3 are outgoing-incoming whereas v_2 and v_4 are incoming-outgoing. The precise values of the **eight others expected phases** in the WKB expansion can be found in [2, paragraph 6.9.1].

As in paragraph 3.4 we are not interested in the construction of the geometric optics expansion but we want to study the behaviour of the rays associated with the phases $(\varphi_j)_{j=1,\ldots,4}$ when T being big.

If we start from a point $(x_0,0)\in\partial\Omega_2$ and make it travel along the characteristics with group velocity v_1, v_2, v_3 and v_4, then after one cycle the ray will hit $\partial\Omega_2$ after a time of travel t_0 in a point $(x_2,0)$. Some computations, like those made in paragraph 3.4, show that, for $x_0>0$ we have :

$$x_{2p}=\beta^{-p}x_0,\ t_p=\widetilde{\alpha}\beta^{-p}x_0,$$

with

$$\beta^{-1}:=\frac{2}{35}=\frac{v_{1,2}}{v_{1,1}}\frac{v_{2,1}}{v_{2,2}}\frac{v_{3,2}}{v_{3,1}}\frac{v_{4,1}}{v_{4,2}},$$

and $\widetilde{\alpha}$ a non-relevant parameter for our purpose. Since $\beta>1$ the ray concentrates at the corner. Moreover the total time of travel towards the corner $\sum_{p\geq 0}t_p$ is the sum of a finite geometric sum so the ray reaches the corner in finite time.

We will come back in paragraph 4.4 on this example, and more precisely on the resolution of the new amplitude equation needed to construct the geometric optics expansion. Let us conclude this paragraph, by Figure 6 that depict the characteristics associated with the group velocities $(v_j)_{j=1,\ldots,4}$:

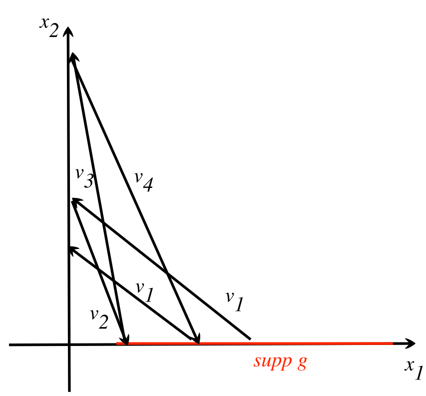

Figure 6: The loop.

4 Geometric optics expansions for selfinteracted trapped rays.

Until the end of this paper, we will study the following hyperbolic corner problem with N equations :

$$\begin{cases} \partial_t u^\varepsilon + A_1\partial_1 u^\varepsilon + A_2\partial_2 u^\varepsilon = 0, \quad (t,x_1,x_2)\in\Omega_T, \\ B_1 u^\varepsilon_{|x_1=0} = g^\varepsilon, \\ B_2 u^\varepsilon_{|x_2=0} = 0, \\ u^\varepsilon_{|t\leq 0} = 0, \end{cases} \tag{18}$$

where, we recall that $A_1, A_2 \in M_N(\mathbb{R})$ with $N\geq 2$, $B_1\in M_{p_1,N}(\mathbb{R})$ and $B_2\in M_{p_2,N}(\mathbb{R})$. Our goal is to construct the WKB approximation to the solution u^ε to (18) when selfinteracting phases occur. But before starting the construction of the geometric optics expansion we shall give a precise and rigorous meaning of the phase generation process described in section 3. This is the object of the following paragraph.

4.1 General framework.

In this paragraph we define a general framework wherein we can construct rigorously geometric optics expansions for corner problems. As already mentioned the geometry of the characteristic variety influences the phase generation process and consequently it also influences the geometric optics expansion. Though not the most general, our framework will be general enough to take into account one loop and selfinteracting phases. Possible extensions are indicated at the end of this article.

4.1.1 Definition of the frequency set and first properties.

Let us start with the definition of what we mean by a frequency set :

Definition 4.1 *Let $\mathscr{I}$ be a subset of $\mathbb{N}$ and $\underline{\tau}\in\mathbb{R}$, $\underline{\tau}\neq 0$. A set indexed by $\mathscr{I}$,*

$$\mathscr{F} := \left\{f_i := (\underline{\tau},\xi_1^i,\xi_2^i), i\in\mathscr{I}\right\},$$

will be a set of frequencies for the corner problem (18) if for all $i\in\mathscr{I}$, f_i satisfies

$$\det\mathscr{L}(f_i) = 0,$$

and one of the following alternatives :
i) $\xi_1^i,\xi_2^i\in\mathbb{R}$.
ii) $\xi_1^i\in(\mathbb{C}\setminus\mathbb{R})$, $\xi_2^i\in\mathbb{R}$ and $Im\ \xi_1^i>0$.
iii) $\xi_2^i\in(\mathbb{C}\setminus\mathbb{R})$, $\xi_1^i\in\mathbb{R}$ and $Im\ \xi_2^i>0$.

In all what follows, if $\mathscr{F}$ is a frequency set for the corner problem (18), we will define :

$$\begin{aligned} \mathscr{F}_{os} &:= \{f_i\in\mathscr{F} \text{ satisfying } i)\}, \\ \mathscr{F}_{ev1} &:= \{f_i\in\mathscr{F} \text{ satisfying } ii)\}, \\ \mathscr{F}_{ev2} &:= \{f_i\in\mathscr{F} \text{ satisfying } iii)\}. \end{aligned}$$

It is clear that the sets $\mathscr{F}_{os}$, $\mathscr{F}_{ev1}$ and $\mathscr{F}_{ev2}$ give a partition of $\mathscr{F}$. Moreover to each $f_i\in\mathscr{F}_{os}$, we can associate a group velocity $v_i := (v_{i,1}, v_{i,2})$. Let us recall that the group velocity v_i is defined in Definition 2.2. The set $\mathscr{F}_{os}$ can be decomposed as follows :

$$\begin{aligned} \mathscr{F}_{ii} &:= \{f_i\in\mathscr{F}_{os}\setminus v_{i,1},v_{i,2}>0\}, \ \mathscr{F}_{io} := \{f_i\in\mathscr{F}_{os}\setminus v_{i,1}>0, v_{i,2}<0\}, \\ \mathscr{F}_{oi} &:= \{f_i\in\mathscr{F}_{os}\setminus v_{i,1}<0, v_{i,2}>0\}, \ \mathscr{F}_{oo} := \{f_i\in\mathscr{F}_{os}\setminus v_{i,1}<0, v_{i,2}<0\}. \\ \mathscr{F}_{g} &:= \{f_i\in\mathscr{F}_{os}\setminus v_{i,1}=0 \text{ or } v_{i,2}=0\}. \end{aligned}$$

The partition of $\mathscr{F}$ induces the following partition of $\mathscr{I}$:

$$\mathscr{I} = \mathscr{I}_g\cup\mathscr{I}_{oo}\cup\mathscr{I}_{io}\cup\mathscr{I}_{oi}\cup\mathscr{I}_{ii}\cup\mathscr{I}_{ev1}\cup\mathscr{I}_{ev2},$$

where we have denoted by $\mathscr{I}_{io}$ (resp. ${}_{g}$, ${}_{oo}$, ${}_{oi}$, ${}_{ii}$, ${}_{ev1}$, ${}_{ev2}$) the set of indeces $i \in \mathscr{I}$ such that the corresponding frequency $f_i \in \mathscr{F}_{io}$ (resp. ${}_{g}$, ${}_{oo}$, ${}_{oi}$, ${}_{ii}$, ${}_{ev1}$, ${}_{ev2}$).

From now on, the source term g^ε on the boundary in (18) reads :

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}(\underline{\tau} t + \underline{\xi}_2 x_2)} g(t, x_2), \tag{19}$$

where the amplitude $g \in H^\infty_f$ and is zero for negative times.

The following definition gives a precise framework for the phase generation process described in section 3. More precisely, this definition qualifies the frequency set that contains all (and only) the frequencies linked with the expected non-zero amplitudes in the WKB expansion of the solution to the corner problem (18).

Definition 4.2 *The corner problem* (18) *is said to be complete for reflections if there exists a set of frequencies $\mathscr{F}$ satisfying the following properties :*

i) $\mathscr{F}$ contains the real roots (in the variable ξ_1) associated with incoming-outgoing or incoming-incoming group velocities and the complex roots with positive imaginary part, to the dispersion equation

$$\det \mathscr{L}(\underline{\tau}, \xi_1, \underline{\xi}_2) = 0.$$

ii) $\mathscr{F}_g = \emptyset$.[3]

iii) If $(\underline{\tau}, \xi_1^i, \xi_2^i) \in \mathscr{F}_{io}$, then $\mathscr{F}$ contains all the roots (in the variable ξ_2), denoted by ξ_2^p, to the dispersion relation $\det \mathscr{L}(\underline{\tau}, \xi_1^i, \xi_2) = 0$, that satisfy one of the following two alternatives :

iii′) $\xi_2^p \in \mathbb{R}$ and the frequency $(\underline{\tau}, \xi_1^i, \xi_2^p)$ is associated with an outgoing-incoming group velocity or an incoming-incoming group velocity.

iii″) Im $\xi_2^p > 0$.

iv) If $(\underline{\tau}, \xi_1^i, \xi_2^i) \in \mathscr{F}_{oi}$, then $\mathscr{F}$ contains all the roots (in the variable ξ_1), denoted by ξ_1^p, to the dispersion relation $\det \mathscr{L}(\underline{\tau}, \xi_1, \xi_2^i) = 0$, that satisfy one of the following two alternatives :

iv′) $\xi_1^p \in \mathbb{R}$ and the frequency $(\underline{\tau}, \xi_1^p, \xi_2^i)$ is associated with an incoming-outgoing or an incoming-incoming group velocity.

iv″) Im $\xi_1^p > 0$.

v) $\mathscr{F}$ is minimal (for the inclusion) for the four preceeding properties.

Remark Point *i)* imposes that the frequency set $\mathscr{F}$ contains all the incoming phases for $\partial\Omega_1$ that are induced by the source term g^ε.

Point *iii)* (resp. *iv)*) explains the generation by reflection on the side $\partial\Omega_2$ (resp. $\partial\Omega_1$) of a wave packet that emanates from the side $\partial\Omega_1$ (resp. $\partial\Omega_2$).

An immediate consequence of the minimality of $\mathscr{F}$ is that $\mathscr{F}_{oo}$ is empty. In all what follows, we will assume that the dispersion relation $\det \mathscr{L}(\underline{\tau}, \xi_1, \underline{\xi}_2) = 0$ has at least one real solution $\underline{\xi}_1$ such that the group velocity for the frequency $\underline{f} := (\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$ is incoming-outgoing. This assumption is, of course, not necessary. However, without this assumption, it is easy to see that the phase generation for the corner problem (18) is not richer than the phase generation for the standard boundary value problem in the half space $\{x_1 \geq 0\}$. Indeed, the minimality of the frequency set $\mathscr{F}$ would imply in this case :

$$\mathscr{F} = \mathscr{F}_{ii} \cup \mathscr{F}_{ev1} \text{ and } \forall f_i \in \mathscr{F},\ \xi_2^i = \underline{\xi}_2.$$

For a corner problem that is complete for reflections, one can define the following applications. These applications are defined on the index set $\mathscr{I}$ and give, in the output, the indeces "in the direct vicinity" of the input index :

$$\Phi,\ \Psi : \mathscr{I} \longrightarrow \mathscr{P}_N(\mathscr{I}),$$

[3] This restriction is probably not necessary. However, for a first work on this subject we did not want to add the technicality induced by the determination of amplitudes associated with glancing frequencies (see [20] for such a construction). Incorporating glancing modes in the WKB expansion is left for future studies.

where $\mathscr{P}_N(\mathscr{I})$ denotes the power set of $\mathscr{I}$ with at most N elements. More precisely, the definitions of Φ and Ψ are : for $i \in \mathscr{I}$, $f_i = (\underline{\tau}, \xi_1^i, \xi_2^i)$,

$$\Phi(i) := \left\{ j \in \mathscr{I} \setminus \xi_2^j = \xi_2^i \right\} \text{ and } \Psi(i) := \left\{ j \in \mathscr{I} \setminus \xi_1^j = \xi_1^i \right\}.$$

Thanks to these applications, the index set $\mathscr{I}$ can be seen as a graph. This graph structure will be more abstract than the desciption of $\mathscr{I}$ based on the wave packet reflections, but it will be easier to handle with when we will construct the WKB expansion. This graph structure is defined by the following relation : two points $i, j \in \mathscr{I}$ are linked by an edge if and only if $i \in \Phi(j)$ or $i \in \Psi(j)$.

In terms of wave packet reflection, the set $\Phi(i)$ (resp. $\Psi(i)$) is the set of all indeces of the phases that are considered in the reflection of the wave packet with phase f_i on $\partial\Omega_2$. Let us stress that the index i is not necessarily the index of an incident ray but can be the index of one of the reflected rays.

It is easy to see that applications Φ and Ψ have the following properties. One can also check that these properties are independent of the concept of "loop" that will be introduced in the following paragraph.

Proposition 4.1 *If the corner problem* (18) *is complete for reflections, then Φ and Ψ satisfy the following properties :*

i) $\forall i \in \mathscr{I}$, $i \in \Psi(i)$, $i \in \Phi(i)$.

ii) $\forall i \in \mathscr{I}$, $\forall j \in \Psi(i)$, $\forall k \in \Phi(i)$ *we have* $\Psi(i) = \Psi(j)$ *and* $\Phi(i) = \Phi(k)$.

iii) $\forall i \in \mathscr{I}$, $\Phi(i) \cap \mathscr{I}_{ev2} = \emptyset$ *and* $\Psi(i) \cap \mathscr{I}_{ev1} = \emptyset$. *And,* $\forall i \in \mathscr{I}_{ev1}, \forall j \in \mathscr{I}_{ev2}$, *we have* $\Psi(i) \subset \mathscr{I}_{ev1}$, $\Phi(i) \subset \mathscr{I}_{ev2}$.

iv) $\forall i \in \mathscr{I}_{os}$, $\#(\Phi(i) \cap \mathscr{I}_{ev1} \cap \mathscr{I}_{io} \cap \mathscr{I}_{ii}) \leq p_1$, *and* $\#(\Psi(i) \cap \mathscr{I}_{ev2} \cap \mathscr{I}_{oi} \cap \mathscr{I}_{ii}) \leq p_2$.

v) $\forall i \in \mathscr{I}$, *we have on one hand* $\forall i_1, i_2 \in \Phi(i)$, $i_1 \neq i_2$ *:*

$$\Phi(i) \cap \Psi(i_1) = \{i_1\} \text{ and } \Psi(i_1) \cap \Psi(i_2) = \emptyset,$$

and on the other hand, $\forall j_1, j_2 \in \Psi(i)$, $j_1 \neq j_2$ *:*

$$\Psi(i) \cap \Phi(j_1) = \{j_1\} \text{ and } \Phi(j_1) \cap \Phi(j_2) = \emptyset.$$

Proof : Points *i)*, *ii)* and *v)* are direct consequences of the definition of the applications Φ and Ψ. Point *iii)* arises from the definition of the frequency set. Finally point *iv)* is a consequence of the block structure lemma (cf. Theorem 2.1).

□

Thanks to applications Φ and Ψ it is easy to define the notion of two linked indeces in the graph structure of $\mathscr{I}$. In terms of wave packet reflections, this notion means that the index i will be linked with the index j if and only if j is obtained from the wave packet associated with i after several reflections. In other terms, we can say that the index i generates the index j, or that i is the "father" of j. The following definition makes this notion more precise :

Definition 4.3 *If* $\underline{i} \in \mathscr{I}_{io}$, *we say that the index* $j \in \mathscr{I}_{io} \cup \mathscr{I}_{ev1}$ *(resp.* $j \in \mathscr{I}_{oi} \cup \mathscr{I}_{ev2}$*) is linked with the index* $\underline{i}$, *if there exists* $p \in 2\mathbb{N}+1$ *(resp.* $p \in 2\mathbb{N}$*) and a sequence of indeces* $\ell = (\ell_1, \ell_2, ..., \ell_p) \in \mathscr{I}^p$ *such that :*

$\alpha')$ $\ell_1 \in \Psi(\underline{i}) \cap \mathscr{I}_{oi}$, $\ell_2 \in \Phi(\ell_1) \in \mathscr{I}_{io}$, ... , $j \in \Phi(\ell_p)$ *(resp.* $j \in \Psi(\ell_p)$*).*

We say that the index $j \in \mathscr{I}_{ii}$ *is linked with the index* $\underline{i}$, *if there is a sequence of indeces* $\ell = (\ell_1, \ell_2, ..., \ell_p) \in \mathscr{I}^p$ *such that :*

$\beta')$ $\ell_1 \in \Psi(\underline{i}) \cap \mathscr{I}_{oi}$, $\ell_2 \in \Phi(\ell_1) \cap \mathscr{I}_{io}, ..., \begin{cases} j \in \Phi(\ell_p), & p \text{ is odd,} \\ j \in \Psi(\ell_p), & p \text{ is even.} \end{cases}$

If $\underline{i} \in \mathscr{I}_{oi}$, we say that the index $j \in \mathscr{I}_{io} \cup \mathscr{I}_{ev1}$ (resp. $j \in \mathscr{I}_{oi} \cup \mathscr{I}_{ev2}$) is linked with the index $\underline{i}$, if there exists $p \in 2\mathbb{N}$ (resp. $p \in 2\mathbb{N}+1$) and a sequence of indeces $\ell = (\ell_1, \ell_2, ..., \ell_p) \in \mathscr{I}^p$ such that :
$\alpha'')$ $\ell_1 \in \Phi(\underline{i}) \cap \mathscr{I}_{io}$, $\ell_2 \in \Psi(\ell_1) \in \mathscr{I}_{oi}$, ... , $j \in \Phi(\ell_p)$ (resp. $j \in \Psi(\ell_p)$).

We say that the index $j \in \mathscr{I}_{ii}$ is linked with the index $\underline{i}$, if there exists a sequence of indeces $\ell = (\ell_1, \ell_2, ..., \ell_p) \in \mathscr{I}^p$ such that :
$\beta'')$ $\ell_1 \in \Phi(\underline{i}) \cap \mathscr{I}_{io}$, $\ell_2 \in \Psi(\ell_1) \cap \mathscr{I}_{oi}, ..., \begin{cases} j \in \Psi(\ell_p), & p \text{ is odd,} \\ j \in \Phi(\ell_p), & p \text{ is even.} \end{cases}$

Finally, if $\underline{i} \in \mathscr{I}_{ii} \cup \mathscr{I}_{ev1} \cup \mathscr{I}_{ev2}$, there is no element of $\mathscr{I}$ linked with $\underline{i}$.

Moreover, we will say that an index $j \in \mathscr{I}$ is linked with the index $\underline{i}$ by a sequence of type H (for "horizontal") (resp. V (for "vertical")) and we will note $i \underset{H}{\rightarrowtail} j$ (resp. $i \underset{V}{\rightarrowtail} j$) if the sequence $(\underline{i}, \ell_1, \ell_2, ..., \ell_p, j)$ satisfies $\alpha'')$ or $\beta'')$ (resp. $\alpha')$ or $\beta')$).

Let us comment a bit this definition. In terms of wave packet reflections, if one fixes an index $\underline{i} \in \mathscr{I}_{io}$, an index j will be linked with the index $\underline{i}$ if j comes from $\underline{i}$ after several reflections. More precisely, the incoming-outgoing ray associated with $\underline{i}$ has hit the side $\partial\Omega_2$, has been reflected in the outgoing-incoming ray associated with the index ℓ_1. Then the ray of index ℓ_1 has hit the side $\partial\Omega_1$, has generated the incoming-outgoing ray associated with the index ℓ_2, this ray has hit the side $\partial\Omega_2$... and so on until the ray associated with the index ℓ_p has generated by reflection the index j.

The distinction of cases based on the group velocity of the index j in the subcase $\alpha')$ considers the fact that a ray associated with an index in $\mathscr{I}_{io} \cup \mathscr{I}_{ev1}$ (resp. $\mathscr{I}_{oi} \cup \mathscr{I}_{ev2}$) can be generated by a ray associated with ℓ_p only during a reflection on the side $\partial\Omega_1$ (resp. $\partial\Omega_2$), or equivalently after an even (resp. odd) number of reflections. Whereas a ray with an incoming-incoming group velocity can be generated by the ray ℓ_p during a reflection on the side $\partial\Omega_1$ or one the side $\partial\Omega_2$. That is the reason why the subcase $\beta')$ differs from the subcase $\alpha')$.

If one rather sees the index set $\mathscr{I}$ with a graph structure, saying that j is linked with $\underline{i}$ is no more than saying that starting from $\underline{i}$ one can reaches the index j by passing through the indeces ℓ_i, with the following rule of travel : if one reaches ℓ_l by following a vertical (resp. horizontal) edge of the graph, then ℓ_{l+1} will be reached by following a horizontal (resp. vertical) edge. A sequence of type H (resp. V) just means that when we start from $\underline{i}$, the first edge is a horizontal (resp. vertical) one.

The following proposition is an immediate consequence of definitions 4.2 and 4.3.

Proposition 4.2 *Let $\mathscr{F}$ be a complete for reflection frequency set indexed by $\mathscr{I}$. Let $\mathscr{I}_0$ be the set of indeces in $\mathscr{I}$ generated by the source term g^ε, that is to say :*

$$\mathscr{I}_0 := \left\{ i \in \mathscr{I}_{io} \cup \mathscr{I}_{ii} \cup \mathscr{I}_{ev1} \setminus \det \mathscr{L}(\underline{\tau}, \xi_1^i, \underline{\xi_2}) = 0 \right\}.$$

Let $\mathscr{I}_{\mathscr{R}}$ be the set of indeces in $\mathscr{I}$ linked with one of the elements of $\mathscr{I}_0$. Then

$$\mathscr{I}_{\mathscr{R}} = \mathscr{I}.$$

Proof : Let $\mathscr{F}_{\mathscr{R}}$ be the set of frequencies indexed by $\mathscr{I}_{\mathscr{R}}$. It is clear that the set $\mathscr{F}_{\mathscr{R}}$ satisfies points $i)$-$iv)$ of definition 4.2. Let us describe the verification of point $iii)$ to be more convincing.

We fix $\underline{i} \in \mathscr{I}_{\mathscr{R}}$, an incoming-outgoing index. Let $\underline{\ell}$ be a sequence that linked $\underline{i}$ to one of the indeces of $\mathscr{I}_0$. Then indeces in $\mathscr{I}_{oi} \cap \Psi(\underline{i})$, $\mathscr{I}_{ii} \cap \Psi(\underline{i})$ and $\mathscr{I}_{ev2} \cap \Psi(\underline{i})$ are linked with an element in $\mathscr{I}_0$ by the sequence $(\underline{\ell}, \underline{i})$. As a consequence, these indeces are in $\mathscr{I}_{\mathscr{R}}$. We just showed that $\mathscr{I}_{\mathscr{R}}$ satisfies point $iii)$ of the definition 4.2.

We now want to show that $\mathscr{I}_{\mathscr{R}} = \mathscr{I}$. By contradiction, we assume that there exists $j \in (\mathscr{I} \setminus \mathscr{I}_{\mathscr{R}})$. Firstly, if $j \in \mathscr{I}_{ev1} \cup \mathscr{I}_{ev2} \cup \mathscr{I}_{ii}$, then the frequency set indexed by $\mathscr{I} \setminus \{j\}$ still satisfies points $i)$-$iv)$ in the definition 4.2. This fact contradicts the minimality of $\mathscr{F}$.

Then, if $j \in \mathscr{I}_{io} \cup \mathscr{I}_{oi}$, we construct the set of indeces linked with j, and we denote this set by $\widetilde{\mathscr{I}}$. Let $\widetilde{\mathscr{F}}$ be the frequency set indexed by $\widetilde{\mathscr{I}}$. The set $(\widetilde{\mathscr{F}} \cup \mathscr{F}_{\mathscr{R}}) \setminus (\widetilde{\mathscr{F}} \cap \mathscr{F}_{\mathscr{R}})$ satisfies points $i) - iv)$ in the definition

4.2 and is strictly included in $\mathscr{F}$ because $j \in (\widetilde{\mathscr{F}} \cap \mathscr{F}_{\mathscr{R}})$. Once more, this fact is incompatible with the minimality of the frequency set $\mathscr{F}$.

$\square$

Proposition 4.2 concludes the description of our formal framework for frequency sets. Let us stress that in this framework we do not assume that the number of phases in the WKB expansion is finite. Assumption "$\#\mathscr{F} < +\infty$" will only be used to make sure that the formal geometric optics expansion constructed in the following paragraphs is relevant, in the sense that the expansion is well-defined and that it does indeed approximate the exact solution. But it will not be used to construct the WKB expansion, at least, at a formal level.

4.1.2 Frequency sets with loops.

As mentioned in the beginning of this section, the aim of all that follows is to construct rigorous geometric optics expansions for corner problem where some amplitudes in the expansion display a selfinteracting phenomenon. To do that, we will need to consider corner problems whose characteristic variety contains a "loop". By loop, we mean that it is possible to find at least four points on the section of the characteristic variety $V \cap \{\tau = \underline{\tau}\}$ such that if we draw the segments linking these points, we obtain a rectangle or a finite "stairway" (cf. paragraph 3.3 and [17, Figure 8]).

Many kinds of loops are possible and few of them lead to a selfinteraction phenomenon. That is why, in all that follows, we will assume that there is a unique loop and that this loop induces a selfinteraction phenomenon. The uniqueness of the loop is probably not a necessary assumption, but it permits to simplify many steps of the proof and to save a lot of combinatorial arguments. We refer to [2, paragraph 6.10] for more details. The different kinds of loops are defined as follows :

Definition 4.4 *Let $i \in \mathscr{I}$, $p \in 2\mathbb{N}+1$ and $\ell = (\ell_1, ..., \ell_p) \in \mathscr{I}^p$ (we stress that elements of ℓ are not necessarily distinct).*
• *We say that the index $i \in \mathscr{I}$ admits a loop if there exists a sequence ℓ satisfying :*

$$\ell_1 \in \Phi(i), \ell_2 \in \Psi(\ell_1), ..., i \in \Psi(\ell_p).$$

• *A loop for an index i is said to be simple if the sequence ℓ does not contain a periodically repeated subsequence.*
• *An index $i \in \mathscr{I}_{io}$ (resp. $i \in \mathscr{I}_{oi}$) admits a selfinteraction loop if i admits a simple loop and if the sequence (i, ℓ, i) is of type V (resp. H) according to Definition 4.3.*

From now on, let us assume that :

Assumption 4.1 *Let* (18) *be complete for the reflections. We assume that the frequency set $\mathscr{F}$ contains a unique loop, of size 3 and that this loop is a selfinteraction loop. More precisely, we want the following properties to be satisfied :*
vi) $\exists (n_1, n_3) \in \mathscr{I}_{io}^2, (n_2, n_4) \in \mathscr{I}_{oi}^2$ *such that*

$$n_4 \in \Psi(n_1),\ n_3 \in \Phi(n_4),\ n_2 \in \Psi(n_3),\ n_1 \in \Phi(n_2).$$

vii) Let $i \in \mathscr{I}$ an index with a loop $\ell = (\ell_1, ..., \ell_p)$. Then $p = 3$ and $\{i, \ell_1, \ell_2, \ell_3\} = \{n_1, n_2, n_3, n_4\}$.

The fact that we restrict our attention to a loop of size 3 is just made to simplify as much as possible the redaction of the proof. However, all the following construction can be generalized to loops with more than 3 elements.

One of the main difficulties induced by the presence of a loop is that the definition of linked indeces does not permit anymore to define a partial order on the frequency set as it can be done in the case $N = 2$. Indeed, if one considers indeces n_1 and n_3 defined in Assumption4.1 then we have $n_1 \underset{V}{\rightarrowtail} n_3$ and $n_3 \underset{V}{\rightarrowtail} n_1$ but $n_1 \neq n_3$. We will see in paragraph 4.2.1 how this new difficulty can be overcome.

We conclude this paragraph by defining what we mean by "trapped" and "selfinteracting" rays.

Definition 4.5 *A ray of the geometric optics expansion is said to be trapped if when we follow its characteristics, we never escape from a compact set.*

A ray of the geometric optics expansion is said to be selfinteracting if when we follow its characteristics, we can find a repeating sequence of group velocities.

So a trapped ray is a ray which will never escape to "infinity". The ray obtained by following the characteristic lines for the indeces (n_1, n_2, n_3, n_4) is a selfinteracting trapped ray. Whereas, the ray describe in paragraph 3.4 is non-selfinteracting trapped ray.

4.1.3 Some definitions and notations.

For $j \in \mathscr{I}_{ev1} \cup \mathscr{I}_{ii} \cup \mathscr{I}_{io}$ (resp. $i \in \mathscr{I}_{ev2} \cup \mathscr{I}_{ii} \cup \mathscr{I}_{oi}$), we denote by $f^j := (\underline{\tau}, \xi_1^j, \xi_j^2)$ the associated frequency. Let us recall that thanks to the uniform Kreiss-Lopatinskii condition, it is possible to define ϕ_1^j (resp. ϕ_2^j) the inverse of B_1 (resp. B_2) restricted to the stable subspace $E_1^s(i\underline{\tau}, \xi_2^j)$ (resp. $E_2^s(i\underline{\tau}, \xi_1^j)$).

To construct the amplitudes in the WKB expansion, we will need the following projectors and partial inverses :

Definition 4.6 *For $j \in \{1, 2\}$, and $f_k = (\underline{\tau}, \xi_1^k, \xi_2^k) \in \mathscr{F}$, let us denote by $P_{s,j}^k$ (resp. $P_{s,j}^k$), the projector on $E_j^{s,e}(i\underline{\tau}, \xi_{3-j}^k)$ (resp. $E_i^{u,e}(i\underline{\tau}, \xi_{3-j}^k)$) associated with the direct sum* (3)*, and P_1^k (resp. P_2^k) the projector on $\ker \mathscr{L}(f_k)$ associated with the sums* (5) *(resp.* (6)*).*

Let us denote by $Q_{s,j}^k$ (resp. $Q_{s,j}^k$), the projector on $E_j^{s,e}(i\underline{\tau}, \xi_{3-j}^k)$ (resp. $E_j^{u,e}(i\underline{\tau}, \xi_{3-j}^k)$) associated with the direct sum (4)*, and Q_j^k (resp. Q_2^k) the projector on $A_1 \ker \mathscr{L}(f_k)$ (resp. $A_2 \ker \mathscr{L}(f_k)$) associated with the sums* (7) *(resp.* (8)*).*

Let R_j^k be the partial of $\mathscr{L}(f_k)$, defined by the two relations

$$R_j^k \mathscr{L}(f_k) = I - P_j^k, \quad P_j^k R_j^k = R_j^k Q_j^k = 0. \tag{20}$$

Finally, to simplify as much as possible the notations, set :

$$S_1^k := P_1^k \phi_1^k, \quad S_2^k := P_2^k \phi_2^k, \quad S_{s,1}^k := P_{s,i}^k \phi_1^k, \quad S_{s,2}^k := P_{s,2}^k \phi_2^k.$$

An important remark is that for $k \in \mathscr{F}_{os}$ if f_k is the associated frequency then : $\mathrm{Ran}\mathscr{L}(f_k) = \ker Q_1^k = \ker Q_2^k$, and that for $j \in \{1, 2\}$, Q_j^k induces an isomorphism from $\mathrm{Ran}P_j^k$ to $\mathrm{Ran}Q_j^k$.

We will have to solve transport equations, so the following variables will be convenient :

$$\forall j \in \mathscr{I}_{io},\ t_{io}^j(t, x_1) := t - \frac{1}{v_{j,1}} x_1,\ \ x_{io}^j(x_1, x_2) := x_2 - \frac{v_{j,2}}{v_{j,1}} x_1, \tag{21}$$

$$\forall j \in \mathscr{I}_{oi},\ t_{oi}^j(t, x_2) := t - \frac{1}{v_{j,2}} x_2,\ \ x_{oi}^j(x_1, x_2) := x_1 - \frac{v_{j,1}}{v_{j,2}} x_2. \tag{22}$$

4.2 Construction of the WKB expansion.

During all the construction, we will have to consider three kinds of phases, namely oscillating phases, evanescent phases for the side $\partial\Omega_1$ and evanescent phases for the side $\partial\Omega_2$. These will be denoted by :

$$\begin{array}{rcl} \varphi_k(t, x) & := & \langle (t, x), f_k \rangle\,, \ \ f_k \in \mathscr{F}_{os}, \\ \psi_{k,1}(t, x_2) & := & \langle (t, 0, x_2), f_k \rangle\,, \ \ f_k \in \mathscr{F}_{ev1} \cup \mathscr{F}_{os}, \\ \psi_{k,2}(t, x_1) & := & \langle (t, x_1, 0), f_k \rangle\,, \ \ f_k \in \mathscr{F}_{ev2} \cup \mathscr{F}_{os}. \end{array}$$

For a given amplitude $g \in H_f^\infty$, zero for negative times, we will work with a source term on the side $\partial\Omega_1$ of the form :

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}(\underline{\tau} t + x_2 \xi_2^{n_1})} g(t, x_2).$$

That is to say a source term that "turns on" the index n_1 on the loop, and that has an incoming group velocity for the side $\partial\Omega_1$. So, we expect that the source term g^ε will generate a wave packet propagating towards the side $\partial\Omega_2$.

As in [9], evanescent modes will be treated in a "monoblock" way. That is to say that for an index $\underline{i} \in \mathscr{I}_{ev1}$ (resp. $\underline{i} \in \mathscr{I}_{ev2}$), all the indeces $j \in \mathscr{I}_{ev1} \cap \Phi(\underline{i})$ (resp. $j \in \mathscr{I}_{ev2} \cap \Psi(\underline{i})$) will contribute to a single vector valued amplitude. To write off the ansatz and to describe with enough precision the boundary conditions, it is useful to introduce the two equivalence relations $\underset{\Phi}{\sim}$ and $\underset{\Psi}{\sim}$ defined by :

$$i \underset{\Phi}{\sim} j \Longleftrightarrow j \in \Phi(i), \text{ and } i \underset{\Psi}{\sim} j \Longleftrightarrow j \in \Psi(i).$$

The fact that these relations are effectively equivalence relations is a direct consequence of Proposition 4.1.

Let $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$) be the set of equivalence classes for the relation $\underset{\Phi}{\sim}$ (resp. $\underset{\Psi}{\sim}$), and $\mathscr{R}_1$ (resp. $\mathscr{R}_2$), be a set of class representative for $\mathfrak{C}_1$ (resp. $\mathfrak{C}_2$). So $\mathscr{R}_1$ (resp. $\mathscr{R}_2$) is a set of indeces which include all the possible values for ξ_2 (resp. ξ_1) of the differents frequencies. Let us define $\mathfrak{R}_1$ and $\mathfrak{R}_2$ by :

$$\mathfrak{R}_1 := \{i \in \mathscr{R}_1 \setminus \Phi(i) \cap \mathscr{I}_{ev1} \neq \emptyset\}, \tag{23}$$
$$\mathfrak{R}_2 := \{i \in \mathscr{R}_2 \setminus \Psi(i) \cap \mathscr{I}_{ev2} \neq \emptyset\}. \tag{24}$$

$\mathfrak{R}_1$ (resp. $\mathfrak{R}_2$) is a set of class representative of the values in ξ_2 (resp. ξ_1) for which there is an evanescent mode for the side $\partial\Omega_1$ (resp. $\partial\Omega_2$). At last, without loss of generality, we can always assume that $n_1 \in \mathscr{R}_2$, in other words, we choose n_1 as a class representative of its equivalence class.

We take for ansatz :

$$\begin{aligned} u^\varepsilon(t,x) \quad &\sim \quad \sum_{k\in\mathscr{I}_{os}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \sum_{n\geq 0} \varepsilon^n u_{n,k}(t,x) \\ &+ \quad \sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}(t,x_2)} \sum_{n\geq 0} \varepsilon^n U_{n,k,1}\left(t,x,\frac{x_1}{\varepsilon}\right) + \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)} \sum_{n\geq 0} \varepsilon^n U_{n,k,2}\left(t,x,\frac{x_2}{\varepsilon}\right). \end{aligned} \tag{25}$$

And we now want to determine the profiles $u_{n,k}$ and $U_{n,k,i}$. We are looking for oscillating profiles $u_{n,k}$ in the space $H^\infty(\Omega_T)$. Whereas, the space for the evanescent profiles is (see [9]) :

Definition 4.7 *For $i = 1,2$, the set $P_{ev,i}$ of evanescent profiles for the side $\partial\Omega_i$ is defined as functions $U(t,x,X_i) \in H^\infty(\Omega_T \times \mathbb{R}_+)$ for which there exists a positive δ such that $e^{\delta X_i} U(t,x,X_i) \in H^\infty(\Omega_T \times \mathbb{R}_+)$.*

Plugging the ansatz (25) in the evolution equation of the corner problem (18) and identifying in terms of powers of ε lead us to solve the cascade of equations :

$$\begin{cases} \mathscr{L}(d\varphi_k)u_{0,k} = 0, & \forall k \in \mathscr{I}_{os}, \\ i\mathscr{L}(d\varphi_k)u_{n+1,k} + L(\partial)u_{n,k} = 0, & \forall n \in \mathbb{N}, \forall k \in \mathscr{I}_{os}, \\ L_k(\partial_{X_1})U_{0,k,1} = 0, & \forall k \in \mathfrak{R}_1, \\ L_k(\partial_{X_1})U_{n+1,k,1} + L(\partial)U_{n,k,1} = 0, & \forall n \in \mathbb{N}, \forall k \in \mathfrak{R}_1, \\ L_k(\partial_{X_2})U_{0,k,2} = 0, & \forall k \in \mathfrak{R}_2, \\ L_k(\partial_{X_2})U_{n+1,k,2} + L(\partial)U_{n,k,2} = 0, & \forall n \in \mathbb{N}, \forall k \in \mathfrak{R}_2, \end{cases} \tag{26}$$

where the "fast" differentiation operators $L_k(\partial_{X_1})$ and $L_k(\partial_{X_2})$ are given by :

$$L_k(\partial_{X_1}) := A_1\big(\partial_{X_1} - \mathscr{A}_1(\underline{\tau}, \xi_2^k)\big), \text{ for } k \in \mathfrak{R}_1,$$
$$L_k(\partial_{X_2}) := A_2\big(\partial_{X_2} - \mathscr{A}_2(\underline{\tau}, \xi_1^k)\big), \text{ for } k \in \mathfrak{R}_2.$$

Then, plugging the ansatz (25) in the boundary conditions on the sides $\partial\Omega_1$ and $\partial\Omega_2$ gives :

$$\begin{aligned} B_1 \left[\sum_{k\in\mathscr{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,1}} u_{n,k}(t,0,x_2) + \sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}} U_{n,k,1}(t,0,x_2,0) \quad + \quad \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,2}\left(t,0,x_2,\frac{x_2}{\varepsilon}\right) \right] \\ = \quad \delta_{n,0} e^{\frac{i}{\varepsilon}\psi_{n_1,1}} g, \end{aligned} \tag{27}$$

and

$$B_2\left[\sum_{k\in\mathscr{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,2}}u_{n,k}(t,x_1,0)+\sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}}U_{n,k,2}(t,x_1,0,0)+\sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\underline{\tau}t}U_{n,k,1}\left(t,x_1,0,\frac{x_1}{\varepsilon}\right)\right]=0. \tag{28}$$

But, thanks to the definition of the evanescent profiles spaces, the functions $U_{n,k,2}\left(t,0,x_2,\frac{x_2}{\varepsilon}\right)$ and $U_{n,k,1}\left(t,x_1,0,\frac{x_1}{\varepsilon}\right)$, which appear in (27) and (28) respectively, are $O(\varepsilon^\infty)$. So, one can rewrite the boundary conditions (27) and (28) under the form :

$$B_1\left[\sum_{k\in\mathscr{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,1}}u_{n,k}(t,0,x_2)+\sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}}U_{n,k,1}(t,0,x_2,0)\right]=\delta_{n,0}e^{\frac{i}{\varepsilon}\psi_{n_1,1}}g(t,x_2), \tag{29}$$

and

$$B_2\left[\sum_{k\in\mathscr{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)}u_{n,k}(t,x_1,0)+\sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)}U_{n,k,2}(t,x_1,0,0)\right]=0. \tag{30}$$

Thanks to the linear independence of the phases, the boundary conditions (29) and (30) can be decomposed as the following cascade of equations :

$$\begin{cases}
B_1\left[\sum_{j\in\Phi(n_1)\cap\mathscr{I}_{os}} u_{n,j}+U_{n,n_1,1_{|X_1=0}}\right]_{|x_1=0}=\delta_{n,0}g, & \forall n\in\mathbb{N},\ \text{if } n_1\in\mathfrak{R}_1,\\
B_1\left[\sum_{j\in\Phi(n_1)} u_{n,j}\right]_{|x_1=0}=\delta_{n,0}g, & \forall n\in\mathbb{N},\ \text{if } n_1\notin\mathfrak{R}_1,\\
B_1\left[\sum_{j\in\Phi(k)\cap\mathscr{I}_{os}} u_{n,j}+U_{n,k,1_{|X_1=0}}\right]_{|x_1=0}=0, & \forall n\in\mathbb{N},\forall k\in\mathfrak{R}_1\setminus\{n_1\},\\
B_1\left[\sum_{j\in\Phi(k)} u_{n,j}\right]_{|x_1=0}=0, & \forall n\in\mathbb{N},\forall k\notin\mathfrak{R}_1\setminus\{n_1\},\\
B_2\left[\sum_{j\in\Psi(k)\cap\mathscr{I}_{os}} u_{n,j}+U_{n,k,2_{|X_2=0}}\right]_{|x_2=0}=0, & \forall n\in\mathbb{N},\forall k\in\mathfrak{R}_2,\\
B_2\left[\sum_{j\in\Psi(k)} u_{n,j}\right]_{|x_2=0}=0, & \forall n\in\mathbb{N},\forall k\notin\mathfrak{R}_2.
\end{cases} \tag{31}$$

At last, plugging the ansatz (25) in the initial condition of the corner problem (18) leads us to solve :

$$\forall n\in\mathbb{N},\quad \begin{cases}
u_{n,k_{|t=0}}=0, & \forall k\in\mathscr{I}_{os},\\
U_{n,k,1_{|t=0}}=0, & \forall k\in\mathfrak{R}_1,\\
U_{n,k,2_{|t=0}}=0, & \forall k\in\mathfrak{R}_2.
\end{cases} \tag{32}$$

The main steps in the construction of the geometric optics expansion are the following. In a first time, before solving the WKB cascade, we will describe a global structure on the set of indeces $\mathscr{I}$. More precisely, this structure is based on a partition which takes into account the different relations that an index can have with the elements of the loop. We will thus be able to express $\mathscr{I}$ as a union of non-intersecting "trees" (or ordered sets by the relations $\underset{H}{\rightarrowtail}$ and $\underset{V}{\rightarrowtail}$, see definition 4.3). Then in a second time, we will construct the amplitudes for the indeces of the loop. To do this, we will need a new invertibility condition, which will be studied in paragraph 4.4.

Thanks to the knowledge of the amplitudes associated with the loop, we will be able to construct the amplitudes in a direct neighborhood of the indeces of the loop. In other terms, the new invertibility condition will be used to start the construction of the geometric optics expansion.

Then, to construct the remaining amplitudes, we will first make a more precise study of the structure of the "trees" that form $\mathscr{I}$. Using this more precise analysis, we will see that the construction of the amplitudes in these "trees" is rather easy because one can define a partial order on these "trees".

The scheme of proof, and more precisely the order of construction of the amplitudes will be exactly the same for higher order terms.

4.2.1 Global structure of the set of indeces $\mathscr{I}$.

In this paragraph we will construct a partition of $\mathscr{I}$ based upon the position of the indeces compared to the loop index n_1 and no more on the different kinds of elements in $\mathscr{I}$. More precisely, the partition will be based upon the different kinds of sequence that can link an index i to the loop index n_1.

The idea of the construction is the following ; firstly, thanks to Proposition 4.2, we know that every index i in $\mathscr{I}$ is linked by a sequence of type V to one of the indeces of $\mathscr{I}_0$ (cf. definition 4.3). Without loss of generality, one can always assume that for all index i the sequence linking i to the index in $\mathscr{I}_0$ does not start by the subsequence (n_4, n_2, n_3, n_1).
The following lemma is also immediate :

Lemma 4.1 *For all $i \in \mathscr{I}$, there exists at least one sequence of type V linking i to n_1. Equivalently, for all $i \in \mathscr{I}$,*

$$n_1 \underset{V}{\rightarrowtail} i,$$

where the notation $\underset{V}{\rightarrowtail}$ has been introduced in definition 4.3.

Proof : It is sufficient to treat the case of indeces i linked with i_0 for $i_0 \in \mathscr{I}_0 \setminus \{n_1\}$. For such indeces, there exists a sequence, denoted by $\widetilde{\ell}$, of type V linking i to i_0. By definition, $i_0 \in \Phi(n_1)$. So i is linked with n_1 by the type V sequence defined by $\ell = (n_4, n_3, n_2, i_0, \widetilde{\ell})$.

$\square$

Now, let $i \in \mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}$, let $\ell^i = (\ell_1, \ell_2, ..., \ell_p)$ be a type V sequence linking i to n_1. The way to construct the sets, denoted $A_{a_l}, B_{b_m}, C_{c_q}, D_{d_r}$, of the sought partition is based on the following algorithm :

Let $\mathtt{C}_1 := \#\Psi(n_1) - 2$, and :

$$\Psi(n_1) \setminus \{n_1, n_4\} := \left\{a_1, a_2, ..., a_{\mathtt{C}_1}\right\}.$$

Let $l \in \left\{1, ..., \mathtt{C}_1\right\}$, we will say that $i \in A_{a_l}$ if and only if the sequence ℓ^i can be chosen such that $\ell_1 = a_l$.

At this stage, we have treated all the sequences that do not start by n_4. To treat the sequences that start by n_4, let $\mathtt{C}_4 := \#\Phi(n_4) - 2$, and

$$\Phi(n_4) \setminus \{n_3, n_4\} := \left\{b_1, b_2, ..., b_{\mathtt{C}_4}\right\}.$$

Then for $m \in \left\{1, ..., \mathtt{C}_2\right\}$, we will say that $i \in B_{b_m}$ if and only if the sequence ℓ^i can be chosen such that $\ell_1 = n_4$ and $\ell_2 = b_m$.
Consequently we have treated all the sequences ℓ^i except those starting by (n_4, n_3).

Finally let $\mathtt{C}_3 := \#\Psi(n_3) - 2$, $\mathtt{C}_2 := \#\Phi(n_2) - 2$ and :

$$\Psi(n_3) \setminus \{n_2, n_3\} := \left\{c_1, c_2, ..., c_{\mathtt{C}_3}\right\}, \ \Phi(n_2) \setminus \{n_1, n_1\} := \left\{d_1, d_2, ..., d_{\mathtt{C}_2}\right\}.$$

We define the sets C_{c_q} and D_{d_r} by the relations :
◇ For $q \in \left\{1, ..., \mathtt{C}_3\right\}$, $i \in C_{c_q}$ if and only if the sequence ℓ^i can be chosen such that $\ell_1 = n_4$, $\ell_2 = n_3$ and $\ell_3 = c_q$.
◇ For $r \in \left\{1, ..., \mathtt{C}_2\right\}$, $i \in D_{d_r}$ if and only if the sequence ℓ^i can be chosen such that $\ell_1 = n_4$, $\ell_2 = n_3$, $\ell_3 = n_2$ and $\ell_4 = d_r$.

This algorithm permits to consider all the possible sequences because no sequence starts by the subsequence (n_4, n_3, n_2, n_1). Then, we repeat this construction for all the potential sequences linking i to n_1.

It is thus clear that

$$(\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}) = \left(\cup_{l \leq \mathtt{C}_1} A_{a_l}\right) \bigcup \left(\cup_{m \leq \mathtt{C}_2} B_{b_m}\right) \bigcup \left(\cup_{q \leq \mathtt{C}_3} C_{c_q}\right) \bigcup \left(\cup_{r \leq \mathtt{C}_4} D_{d_r}\right). \tag{33}$$

The sets A_{a_l} and B_{b_m} can be characterized as follows : A_{a_l} is the set of indeces $i \in \mathscr{I}$ such that $a_l \underset{H}{\rightarrowtail} i$, whereas B_{b_m} is the set of indeces $i \in \mathscr{I}$ such that $b_m \underset{V}{\rightarrowtail} i$. In terms of wave packet reflection, the set A_{a_l} gathers the indeces obtained by reflection of the phase associated with the index a_l, this phase being obtained by reflection of the wave packet associated with n_1 on the side $\partial\Omega_2$. In a similar way, B_{b_m} gathers the indeces obtained by reflection of the phase associated with the index b_j. The phase associated with b_j being obtained by reflection of the phase associated with n_4 on the side $\partial\Omega_1$. An analogous characterization stands for the sets C_{c_q} and D_{d_r}.

Lemma 4.2 *The decompostion*

$$(\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}) = \left(\cup_{l \le \mathtt{C}_1} A_{a_l}\right) \bigcup \left(\cup_{m \le \mathtt{C}_2} B_{b_m}\right) \bigcup \left(\cup_{n \le \mathtt{C}_3} C_{c_n}\right) \bigcup \left(\cup_{q \le \mathtt{C}_4} D_{d_q}\right),$$

is a partition of $\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}$.

Proof : Let us first define the "mirror" sequence of a sequence by the following relation :

$$\forall \ell = (\ell_1, \ell_2, ..., \ell_p) \in \mathscr{I}^p, \quad \overline{\ell} := (\ell_p, \ell_{p-1}, ..., \ell_1) \in \mathscr{I}^p.$$

Let $l, l' \in \{1, ..., \mathtt{C}_1\}$, $l \neq l'$.
$\diamond$ Proof of $A_{a_l} \cap A_{a_{l'}} = \emptyset$:
We argue by contradiction. Let us assume that there exists $i \in A_{a_l} \cap A_{a_{l'}}$. Then by definition, there exists a type H sequence $\ell = (\ell_1, ..., \ell_p)$ linking i to a_l and a type H sequence $\ell' = (\ell'_1, ..., \ell'_{p'})$ linking i to $a_{l'}$. We now have to consider several cases depending on the oddness/evenness of p and p'.

$\bullet p, p' \in 2\mathbb{N}$.
By definition of H type sequences, we have $i \in \Psi(\ell_p)$ and $i \in \Psi(\ell'_{p'})$. Thanks to point $ii)$ of Proposition 4.1, $\ell'_{p'} \in \Psi(\ell_p)$. The sequence $(\ell, \overline{\ell'})$ is consequently a type H sequence linking a_l to $a_{l'}$. But $a_l \in \Phi(a_{l'})$, so the sequence $(\ell, \overline{\ell'}, a_{l'})$ is a loop for the index a_l with exactly $p + p' + 1$ elements. This contradicts Assumption 4.1.

$\bullet p \in 2\mathbb{N}, p' \in 2\mathbb{N} + 1$.
Now $i \in \Psi(\ell_p)$ and $i \in \Phi(\ell'_{p'})$ or equivalenty $\ell'_{p'} \in \Phi(i)$. The sequence $(\ell, i, \overline{\ell'})$ is a type H sequence linking a_l to $a_{l'}$. Then $(\ell, i, \overline{\ell'}, a_{l'})$ is a loop for the index a_l with $p + p' + 2$ elements. Once again, it contradicts Assumption 4.1.
The case $p, p' \in 2\mathbb{N} + 1$ is quite similar to the case $p, p' \in 2\mathbb{N}$, so we omit the proof.

We now deal with the proof of the property $A_{a_l} \cap B_{b_m} = \emptyset$, the other proofs showing that the other kinds of intersection are empty are analogous and consequently they will not be treated here.
$\diamond$ Proof of $A_{a_l} \cap B_{b_m} = \emptyset$:
Once again, we argue by contradiction. Let $i \in A_{a_l} \cap B_{b_m}$. Then by definition of the sets A_{a_l} and B_{b_m}, $a_l \underset{H}{\rightarrowtail} i$ and $b_m \underset{V}{\rightarrowtail} i$. That is to say that there exists $\ell = (\ell_1, ..., \ell_p)$ a type H sequence linking i to a_l and a type V sequence $\ell' = (\ell'_1, ..., \ell'_{p'})$ linking i to b_m. We have to consider the following cases :

$\bullet p, p' \in 2\mathbb{N}$.
We have $i \in \Psi(\ell_p)$ and $i \in \Phi(\ell'_{p'})$. So it is possible to show exactly as in the proof of one of the above subcases that the sequence $(\ell, i, \overline{\ell'})$ links a_l to b_m. It follows from $a_l \in \Psi(n_2)$ and $n_4 \in \Phi(b_m)$, that the sequence $(\ell, i, \overline{\ell'}, b_m, n_4)$ is a loop for the index a_l with an odd number of elements.

$\bullet p \in 2\mathbb{N}, p' \in 2\mathbb{N} + 1$.
We show that the sequence $(\ell, \overline{\ell'})$ links a_l to b_m. So $(\ell, \overline{\ell'}, b_m, n_4)$ is a loop for a_l with an odd number of elements, which is again a contradiction with Assumption 4.1.

□

We have just shown that

$$\left(\cup_{i \le \mathtt{C}_1} A_{a_i}\right) \bigcup \left(\cup_{i \le \mathtt{C}_2} B_{b_i}\right) \bigcup \left(\cup_{i \le \mathtt{C}_3} C_{c_i}\right) \bigcup \left(\cup_{i \le \mathtt{C}_4} D_{d_i}\right), \tag{34}$$

is a partition of $\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}$. A consequence is that to determine all the amplitudes in the WKB expansion, it will be sufficient to construct the amplitude for the indeces on the loop and then the amplitudes in each set of the partition (34).

Moreover, the construction of the amplitudes in each set of the partition (34) can be made intrinsically in this set. Indeed, the fact that (34) is a partition implies that an index i in one set of (34) is only linked, by the boundary conditions (31), with other indeces in the same set.

A last consequence of the fact that (34) is a partition of the frequency set is the following refinement of Proposition 4.1 :

Proposition 4.3 *Let* (18) *be complete for the reflections, under Assumption 4.1. Let* $\mathscr{I}$ *be the index set ; then* Φ *and* Ψ *satisfy, in addition to the properties of Proposition 4.1, the two extra properties :*

viii)

$$\begin{array}{lcl} \Phi(n_1)\setminus\{n_2\}\subset\mathscr{I}_{io}\cup\mathscr{I}_{ii}\cup\mathscr{I}_{ev1} & , & \Psi(n_1)\setminus\{n_1\}\subset\mathscr{I}_{oi}\cup\mathscr{I}_{ii}\cup\mathscr{I}_{ev2}, \\ \Phi(n_4)\setminus\{n_4\}\subset\mathscr{I}_{io}\cup\mathscr{I}_{ii}\cup\mathscr{I}_{ev1} & , & \Psi(n_3)\setminus\{n_3\}\subset\mathscr{I}_{oi}\cup\mathscr{I}_{ii}\cup\mathscr{I}_{ev2}. \end{array}$$

ix) Let $i\in\mathscr{I}_{ii}\cup\mathscr{I}_{ev1}$ *and* $j\in\mathscr{I}_{ii}\cup\mathscr{I}_{ev2}$ *then*

$$\begin{array}{lcl} i\in\Phi(n_1)\implies\Psi(i)=\{i\} & , & j\in\Psi(n_1)\implies\Phi(j)=\{j\}\,, \\ i\in\Phi(n_4)\implies\Psi(i)=\{i\} & , & j\in\Psi(n_3)\implies\Phi(j)=\{j\}\,. \end{array}$$

Proof : $\diamond$ Proof of *viii)* :

We will here just show the first assertion that is to say : $\Phi(n_1)\setminus\{n_2\}\subset\mathscr{I}_{io}\cup\mathscr{I}_{ii}\cup\mathscr{I}_{ev1}$. By contradiction, let $i\in\Phi(n_1)\cap\mathscr{I}_{oi}$, $i\neq n_2$. Then, there exists $j\in\Psi(i)\cap\mathscr{I}_{io}$, otherwise the frequency set indexed by $\mathscr{I}\setminus\{i\}$ is strictly included in $\mathscr{F}$ and satisfies $i)-iv)$ of Definition 4.2.

Thanks to Lemma 4.1, we know that there exists a type V sequence, $\ell=(\ell_1,\ell_2,...,\ell_p)$, with necessarily $p\in 2\mathbb{N}+1$ (because $n_1,j\in\mathscr{I}_{io}$, see definition 4.3), such that $n_1\underset{V}{\rightarrowtail}j$. The sequence (ℓ,j,i) is a selfinteracting loop for n_1 with an odd number of elements, but it is not the same loop as $\{n_1,n_2,n_3,n_4\}$. This is a contradiction with Assumption 4.1.

$\diamond$ Proof of *ix)* :

The proof of *ix)* uses exactly the same reasoning as for *viii)* and we will omit it here. The only difference is that we can not conclude that the loop is a selfinteracting one because it may contain indeces in $\mathscr{I}_{ii}$. Then we need the uniqueness assumption of a loop and not only the uniqueness assumption of a selfinteracting loop.

□

Property $\Phi(n_1)\setminus\{n_2\}\subset\mathscr{I}_{io}\cup\mathscr{I}_{ii}\cup\mathscr{I}_{ev1}$ of point *viii)* in Proposition 4.3 means that even if the characteristic variety contains a loop, all the frequencies (but n_2) associated with outgoing-incoming group velocity are initially discarded. We already justify this observation in the phase generation process described in section 3. Point *ix)* means that, thanks to the uniqueness assumption of a loop, an incoming-incoming phase in the direct neighborhood of the loop can only be generated by reflection on one single side of $\partial\Omega$ and not on both sides.

Thanks to Proposition 4.3, partition (34) can be rewritten under the form :

$$\begin{array}{rcl} (\mathscr{I}\setminus\{n_1,n_2,n_3,n_4\}) & = & \left(\cup_{a_l\in\Psi(n_1)\cap\mathscr{I}_{io}}A_{a_l}\ \cup_{a_l\in\Psi(n_1)\cap(\mathscr{I}_{ev1}\cup\mathscr{I}_{ii})}\{a_l\}\right) \\ & \bigcup & \left(\cup_{b_m\in\Phi(n_4)\cap\mathscr{I}_{oi}}B_{b_m}\ \cup_{b_m\in\Phi(n_4)\cap(\mathscr{I}_{ev2}\cup\mathscr{I}_{ii})}\{b_m\}\right) \\ & \bigcup & \left(\cup_{c_q\in\Psi(n_3)\cap\mathscr{I}_{io}}C_{c_q}\ \cup_{c_q\in\Psi(n_3)\cap(\mathscr{I}_{ev1}\cup\mathscr{I}_{ii})}\{c_q\}\right) \\ & \bigcup & \left(\cup_{d_r\in\Phi(n_2)\cap\mathscr{I}_{io}}D_{d_r}\ \cup_{d\in\Phi(n_2)\cap(\mathscr{I}_{ev2}\cup\mathscr{I}_{ii})}\{d_r\}\right). \end{array} \tag{35}$$

Let us conclude this paragraph by the Figure 7 which illustrates the "tree structure" of the frequency set $\mathscr{F}$:

4.2.2 Determination of the amplitudes on the loop and invertibility condition.

Now that the global structure of the frequency set is described, and thanks to the new properties of applications Φ and Ψ, it is time to start the construction of the amplitudes in the WKB expansion. A good (and natural) choice to initialize this construction is to determine first the amplitudes associated with the loop indeces. To do that, a new amplitude equation will be derived (see paragraph 3.5).

The cascades of equations (26)-(31) and (32) written for $n=0$ and $k=n_1$, tell us that the amplitude u_{0,n_1} satisfies :

$$\left\{\begin{array}{l} \mathscr{L}(d\varphi_{n_1})u_{0,n_1}=0, \\ i\mathscr{L}(d\varphi_{n_1})u_{1,n_1}+L(\partial)u_{0,n_1}=0, \end{array}\right. \tag{36}$$

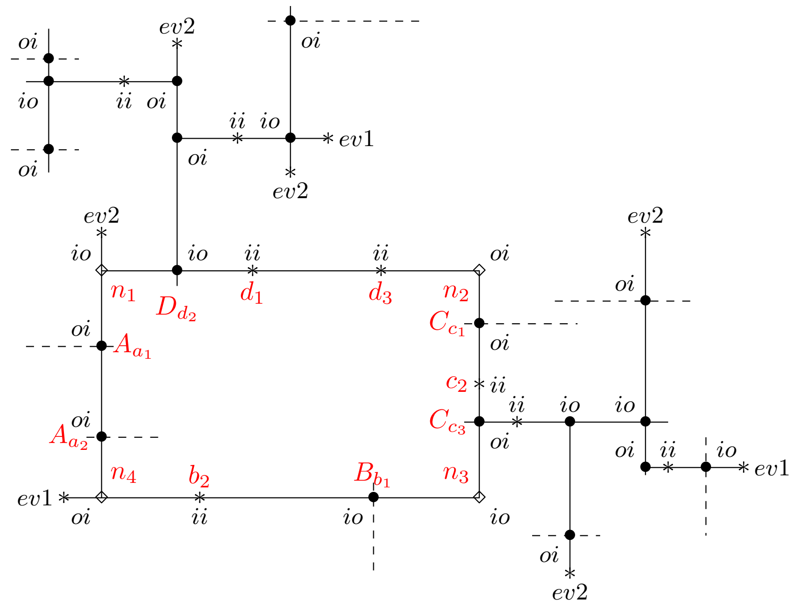

Figure 7: "Tree structure" of the frequency set $\mathscr{F}$.

in the interior, the boundary conditions :

$$\begin{cases} B_1 \left[\sum_{j\in\Phi(n_1)\cap\mathscr{I}_{os}} u_{0,j} + U_{0,n_1,1_{|X_1=0}}\right]_{|x_1=0} = g, & \text{if } n_1 \in \mathfrak{R}_1, \\ B_1 \left[\sum_{j\in\Phi(n_1)} u_{0,j}\right]_{|x_1=0} = g, & \text{if } n_1 \in \mathscr{R}_1 \setminus \mathfrak{R}_1, \end{cases} \tag{37}$$

and

$$\begin{cases} B_2 \left[\sum_{j\in\Psi(n_1)\cap\mathscr{I}_{os}} u_{0,j} + U_{0,n_1,1_{|X_1=0}}\right]_{|x_2=0} = 0, & \text{if } n_1 \in \mathfrak{R}_2, \\ B_2 \left[\sum_{j\in\Psi(n_1)} u_{0,j}\right]_{|x_2=0} = 0, & \text{if } n_1 \in \mathscr{R}_2 \setminus \mathfrak{R}_2, \end{cases} \tag{38}$$

and finally the initial condition :

$$u_{0,n_1}{}_{|t\leq 0} = 0. \tag{39}$$

We will now explain the method of resolution of equations (36)-(37)-(38) and (39). The ideas described below are classical, they explain why the amplitudes associated with oscillating phases satisfy transport equations in the example of paragraph 3.5 and they will be applied to all the oscillating amplitudes.

Firstly, let us remark that the first equation of (36) tells us that the amplitude $u_{0,n_1} \in \ker \mathscr{L}(d\varphi_{n_1})$. In other words, we have the so-called polarization condition :

$$P_1^{n_1} u_{0,n_1} = u_{0,n_1},$$

where $P_1^{n_1}$ is the projector defined in definition 4.6. Now, composing the second equation of (36) by the projector $Q_1^{n_1}$ defined in Definition 4.6 and using the polarization condition give us :

$$Q_1^{n_1} L(\partial) P_1^{n_1} u_{0,n_1} = 0.$$

But Lax lemma [8] tells us that if the corner problem (18) is constantly hyperbolic then we have the following relation :

$$Q_1^{n_1} L(\partial) P_1^{n_1} = (\partial_t + v_{n_1}\cdot\nabla_x) Q_1^{n_1} P_1^{n_1},$$

where v_{n_1} is the group velocity associated with the phase φ_{n_1}. So, the amplitude u_{0,n_1} satisfies the transport equation :

$$(\partial_t + v_{n_1}\cdot\nabla_x)Q_1^{n_1}u_{0,n_1} = 0.$$

We are now interested in the boundary conditions. As mentionned in section 3, the boundary conditions needed to solve a transport equation in a quarter space are linked with the nature of the transport velocity. Let us recall the four possible alternatives :

$\diamond$ The transport velocity is outgoing-outgoing, then no boundary condition has to be imposed.

$\diamond$ The transport velocity is incoming-outgoing, then the transport equation needs a boundary condition on $\partial\Omega_1$ only.

$\diamond$ The transport velocity is outgoing-incoming, then the transport equation needs a boundary condition on $\partial\Omega_2$ only.

$\diamond$ The transport velocity is incoming-incoming, then the transport equation needs a boundary condition on $\partial\Omega_1$ and on $\partial\Omega_2$.

Here, by assumption we have $n_1 \in \mathscr{I}_{io}$, so no boundary condition on $\partial\Omega_2$ has to be imposed and we only keep the boundary condition on $\partial\Omega_2$. As a consequence the amplitude u_{0,n_1} satisfies the transport equation :

$$\begin{cases} (\partial_t + v_{n_1}\cdot\nabla_x)Q_1^{n_1}u_{0,n_1} = 0, \\ B_1\left[\sum_{k\in\Phi(n_1)} u_{0,k}\right]_{|x_1=0} = g, \quad \text{if } n_1 \notin \mathfrak{R}_1, \\ u_{0,n_1}{}_{|t\leq 0} = 0, \end{cases} \tag{40}$$

and,

$$\begin{cases} (\partial_t + v_{n_1}\cdot\nabla_x)Q_1^{n_1}u_{0,n_1} = 0, \\ B_1\left[\sum_{k\in\Phi(n_1)} u_{0,k} + U_{0,n_1,1}{}_{|X_1=0}\right]_{|x_1=0} = g, \quad \text{if } n_1 \in \mathfrak{R}_1. \\ u_{0,n_1}{}_{|t\leq 0} = 0, \end{cases} \tag{41}$$

In both cases, using the fact that $\Phi(n_1)\cap\mathscr{I}_{oi} = \{n_2\}$ thanks to $vi)$ of Assumption 4.1 and $viii)$ of Proposition 4.3, the boundary condition of (40) reads :

$$u_{0,n_1}{}_{|x_1=0} + \sum_{k\in(\Phi(n_1)\cap(\mathscr{I}_{io}\cup\mathscr{I}_{ii}))\setminus\{n_1\}} u_{0,k}{}_{|x_1=0} = \phi_1^{n_1}\left[g - B_1 u_{0,n_2}{}_{|x_1=0}\right],$$

when $n_1 \notin \mathfrak{R}_1$, and

$$u_{0,n_1}{}_{|x_1=0} + \sum_{k\in(\Phi(n_1)\cap(\mathscr{I}_{io}\cup\mathscr{I}_{ii}))\setminus\{n_1\}} u_{0,k}{}_{|x_1=0} + U_{0,n_1,1}{}_{|x_1=X_1=0} = \phi_1^{n_1}\left[g - B_1 u_{0,n_2}{}_{|x_1=0}\right],$$

when $n_1 \in \mathfrak{R}_1$. Multiplying these conditions by the projector $P_1^{n_1}$, using the fact that the $u_{0,k}$ are polarized on $\ker\mathscr{L}(d\varphi_k)$, we obtain, in both cases, that the trace u_{0,n_1} on $\partial\Omega_1$ is given by :

$$u_{0,n_1}{}_{|x_1=0} = S_1^{n_1}\left[g - B_1 u_{0,n_2}{}_{|x_1=0}\right],$$

where we recall that the matrix $S_1^{n_1}$ has been introduced in definition 4.6.

It is now easy to integrate the equation (40) along the characteristics. We obtain the expression of u_{0,n_1} according to its trace on $\partial\Omega_1$; more precisely :

$$u_{0,n_1}(t,x) = S_1^{n_1}\left[g - B_1 u_{0,n_2}{}_{|x_1=0}\right]\left(t_{es}^{n_1}(t,x_1), x_{es}^{n_1}(x_1,x_2)\right),$$

where the new variables $t_{io}^{n_1}$ and $x_{io}^{n_1}$ are defined in (21). As a consequence the trace of u_{0,n_1} on $\partial\Omega_2$ reads :

$$u_{0,n_1}(t,x_1,0) = S_1^{n_1}\left[g - B_1 u_{0,n_2}{}_{|x_1=0}\right]\left(t_{io}^{n_1}(t,x_1), -\frac{v_{n_1,2}}{v_{n_1,1}}x_1\right). \tag{42}$$

Then we can repeat exactly the same reasoning for the second element n_2 of the loop. Indeed, using the fact that $n_2 \in \mathscr{I}_{oi}$, u_{0,n_2} will be determined by integration along the characteristics from its trace on $\partial\Omega_2$. Thanks to Assumption 4.1 and point *viii*) of Proposition 4.3, the trace $u_{0,n_2}{}_{|x_2=0}$ will depend only of the trace $u_{0,n_3}{}_{|x_2=0}$. The trace $u_{0,n_2}{}_{|x_1=0}$ which appears in (42), is consequently given by :

$$u_{0,n_2}(t,0,x_2) = -S_2^{n_2} B_2 u_{0,n_3}{}_{|x_2=0}\left(t_{oi}^{n_2}(t,x_2), -\frac{v_{n_2,1}}{v_{n_2,2}}x_2\right). \tag{43}$$

At last, repeating the same method, we obtain the traces of the two remaining amplitudes for the indeces of the loop :

$$u_{0,n_3}(t,x_1,0) = -S_1^{n_3} B_1 u_{0,n_4}{}_{|x_1=0}\left(t_{io}^{n_3}(t,x_1), -\frac{v_{n_3,2}}{v_{n_3,1}}x_1\right). \tag{44}$$

and

$$u_{0,n_4}(t,0,x_2) = -S_2^{n_4} B_2 u_{0,n_1}{}_{|x_2=0}\left(t_{oi}^{n_4}(t,x_2), -\frac{v_{n_4,1}}{v_{n_4,2}}x_2\right). \tag{45}$$

An important point in this analysis is that at each step of the computation, there is one and only one outgoing phase coupled with the incoming phases in the equivalence classes, for the relations $\underset{\Phi}{\sim}$ and $\underset{\Psi}{\sim}$, of the indeces n_j. This fact will, *a priori*, not be true anymore if one considers a frequency set containing several loops.

Thus, combining equations (42)-(43)-(44)-(45) we obtain, after some computations, the functional equation determining the trace $u_{0,n_1}{}_{|x_2=0}$:

$$(I - \mathbb{T})u_{0,n_1}{}_{|x_2=0} = S_1^{n_1} g\left(t - \frac{1}{v_{n_1,1}}x_1, -\frac{v_{n_1,2}}{v_{n_1,1}}x_1\right), \tag{46}$$

where $\mathbb{T}$ is the operator defined by :

$$(\mathbb{T}w)(t,x_1) \quad := \quad Sw\left(t+\alpha x_1, \beta x_1\right), \tag{47}$$

with :

$$\begin{aligned} S &:= S_1^{n_1} B_1 S_2^{n_2} B_2 S_1^{n_3} B_1 S_2^{n_4} B_2, \\ \alpha &:= \frac{1}{v_{n_1,1}}\left[-1 + \frac{v_{n_1,2}}{v_{n_2,2}} - \frac{v_{n_1,2}v_{n_2,1}}{v_{n_2,2}v_{n_3,1}} + \frac{v_{n_1,2}v_{n_2,1}v_{n_3,2}}{v_{n_2,2}v_{n_3,1}v_{n_4,2}}\right] < 0, \\ \beta &:= \frac{v_{n_4,1}}{v_{n_4,2}}\frac{v_{n_3,2}}{v_{n_3,1}}\frac{v_{n_2,1}}{v_{n_2,2}}\frac{v_{n_1,2}}{v_{n_1,1}} > 0. \end{aligned} \tag{48}$$

Given equation (46), we make the following assumption :

Assumption 4.2 *For all $\gamma > 0$, the operator $(I - \mathbb{T})$ defined in (47) is invertible from $L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)$ to $L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)$, uniformly with respect with the parameter $\gamma > 0$.*

However, for $T > 0$ and for a source term in $L^2(]-\infty,T]\times\mathbb{R}_+)$ zero for negative times, this assumption will only give us amplitudes for indeces of the loop which are $L^2(\Omega_T)$. This is not sufficient to construct the amplitudes of high order in the WKB expansion nor to make sure that the amplitudes linked with an incoming-incoming group velocity are $H^1(\Omega_T)$. We thus need to reinforce Assumption 4.2 in the following way :

Assumption 4.3 *Let $2 \le K \le \infty$, for all $\gamma > 0$, the operator $(I - \mathbb{T})$ defined in (47) is invertible from $H^K_{f,\gamma}$ to $H^K_{f,\gamma}$ uniformly with respect with $\gamma > 0$.*

Let us stress that Assumption 4.3 is (at this stage of the analysis) purely formal and is introduced to construct the WKB expansion. We will show in paragraph 4.4 the following proposition :

Proposition 4.4 *If $|S| < \sqrt{\beta}$ (where S and β are defined in* (48)*), for all $\gamma > 0$, the operator $(I - \mathbb{T})$ is uniformly invertible from $L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$ to $L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$. In particular, for all $T > 0$, the equation* (46) *admits a unique solution $u \in L^2(]-\infty, T] \times \mathbb{R}_+)$, zero for negative times, if the source term $G \in L^2(\partial\Omega_{1,T})$ and is zero for negative times.*

If $\beta \leq 1$ and $G \in H^\infty_f$, under the assumption $|S| < \sqrt{\beta}$, the solution u of the equation $(I - \mathbb{T})u = G$ is in H^∞_f.

If $\beta > 1$ let $K \in \mathbb{N}$ and $G \in H^K_f$; then under the assumption $|S|\beta^{K-\frac{1}{2}} < 1$, the solution u of $(I - \mathbb{T})u = G$ is in H^K_f.

Assumption $|S| < \sqrt{\beta}$, or $|S|\beta^{K-\frac{1}{2}} < 1$, gives us a framework in which we can, firstly ensure enough regularity to construct (at least up to a finite order) the amplitudes in the WKB expansion, secondly construct the incoming-incoming amplitudes. More details and comments about the condition $|S| < \sqrt{\beta}$ will be given in paragraph 4.4.

From now on we denote by $K \in \mathbb{N} \cup \{+\infty\}$ the largest integer such that the solution u of the equation $(I - \mathbb{T})u = G$ is H^K_f, for $G \in H^K_f$. In view of constructing the first corrector term and to ensure that the WKB expansion is a good approximation to the exact solution, we need $K \geq 3$.

From all these considerations about equation (46), it follows that the trace $u_{0,n_1}{}_{|x_2=0}$ is uniquely determined in H^K_f by the formula :

$$u_{0,n_1}(t, x_1, 0) = (I - \mathbb{T})^{-1} S_1^{n_1} g \left(t - \frac{1}{v_{n_1,1}} x_1, -\frac{v_{n_1,2}}{v_{n_1,1}} x_1 \right), \tag{49}$$

an equation which enables to construct the amplitudes u_{0,n_j}, $j = 1, ..., 4$ by using (45), (44) and (43) and integrating along the corresponding characteristics.

We summarize up this construction of the amplitudes associated with loop indeces by the following proposition :

Proposition 4.5 *Under Assumptions 2.1-2.2 on the complete for reflections corner problem* (18)*, under Assumptions 4.1 and 4.3, then for $j = 1, ...4$ and for all $T > 0$, there exist functions $u_{0,n_j} \in H^K(\Omega_T)$, with traces in H^K_f, satisfying the cascades of equations* (26)-(31) *and* (32) *written for $n = 0$ and $k = n_j$.*

4.2.3 Determination of the amplitudes in the direct neighborhood of the loop.

In this paragraph we will show that the knowledge of the amplitudes on the loop and the global structure of the index set $\mathscr{I}$ described in paragraph 4.2.1 are sufficient to construct the amplitudes in the direct neighborhood of the indeces of the loop.

We have chosen to separate this construction from the construction of the amplitudes linked with indeces in the different sets of the partition (35). This choice is motivated by the following two reasons. Firstly we think that it is important to make the computations explicitely at least once (mainly because we do not have described yet the construction for evanescent phases). Secondly, because the construction in the close neighborhood of the loop remains unchanged, instead of the determination of the amplitudes associated with indeces in the different sets of the partition (35), under a weaker uniqueness assumption of the loop (*i.e.* an assumption imposing the uniqueness of a selfinteraction loop, but which allows other types of loops in the frequency set).

We will here describe the determination of the amplitudes in $\Phi(n_4)$, the construction for amplitudes in $\Psi(n_4)$, $\Phi(n_2)$ or $\Psi(n_2)$ is exactly the same. Using point *viii)* of Proposition 4.3, we know that $\Phi(n_4) \cap \mathscr{I}_{oi} = \{n_4\}$. The boundary condition (31) writen for $k = n_4$ reads (if we choose n_4 as a class representative of its own equivalence class for the relation $\underset{\Phi}{\sim}$):

$$B_1 \left[\sum_{j \in \Phi(n_4) \cap (\mathscr{I}_{io} \cup \mathscr{I}_{ii})} u_{0,j} + U_{0,n_4,1|X_1=0} \right]_{|x_1=0} = -B_1 u_{0,n_4|x_1=0}, \text{ if } n_4 \in \mathfrak{R}_1, \tag{50}$$

$$B_1 \left[\sum_{j \in \Phi(n_4) \cap (\mathscr{I}_{io} \cup \mathscr{I}_{ii})} u_{0,j} \right]_{|x_1=0} = -B_1 u_{0,n_4|x_1=0}, \text{ if } n_4 \in \mathscr{R}_1 \setminus \mathfrak{R}_1, \tag{51}$$

where in both cases, the source term is a known element of H_f^K.

Applying the uniform Kreiss-Lopatinskii condition to equations (50) and (51), and composing by the projectors P_1^j for $j \in \Phi(n_4) \cap (\mathscr{I}_{io} \cup \mathscr{I}_{ii})$, and/or by $P_{s,1}^{n_4}$, leads us to solve the uncoupled boundary conditions :

$$\begin{aligned} \forall j \in \Phi(n_4) \cap (\mathscr{I}_{io} \cup \mathscr{I}_{ii}),\ u_{0,j_{|x_1=0}} &= -S_1^j B_1 u_{0,n_{4|x_1=0}}, & (52)\\ \text{if moreover } n_4 \in \mathfrak{R}_1,\ U_{0,n_4,1_{|X_1=x_1=0}} &= -S_{s,1}^j B_1 u_{0,n_{4|x_1=0}}. & (53) \end{aligned}$$

Thus, the construction of the possible evanescent amplitude for the side Ω_1 can be made independently of the construction of the amplitudes for oscillating phases.

Let us first briefly recall how to determine amplitudes for oscillating phases, we have several cases to take into account.

$\diamond\ j \in \mathscr{I}_{io}$

Lax Lemma and the polarization condition enable us to show that the amplitude $u_{0,j}$ satisfies a transport equation with an incoming-outgoing velocity v_j. That is why to construct this amplitude we just need to know its trace on $\partial\Omega_1$. This trace is determined by (52), so integrating along the characteristics, $u_{0,j}$ is given by :

$$u_{0,j}(t,x) = S_1^j B_1 u_{0,n_{4|x_1=0}}\big(t_{io}^j(t,x_1), x_{io}^j(x_1,x_2)\big). \tag{54}$$

An important point for the end of the proof (more specifically for the construction of incoming-incoming amplitudes in the set B_{b_j}) is that $u_{0,j} \in H^K(\Omega_T)$ for all $T > 0$ and that its trace on the side $\partial\Omega_2$ is H_f^K. We can easily see this fact on the formula (54).

In other words, the flatness at the corner of the source term g^ε is transmitted to the amplitudes close to the loop.

$\diamond\ j \in \mathscr{I}_{ii}$

In that case $u_{0,j}$ is solution to a transport equation with incoming-incoming velocity, so its determination needs the traces on both sides $\partial\Omega_1$ and $\partial\Omega_2$. The boundary condition (52) gives the trace on $\partial\Omega_1$. Concerning the trace on $\partial\Omega_2$, point $ix)$ of Proposition 4.3 shows that j is the only element in its equivalence class for the relation $\underset{\Psi}{\sim}$, in particular $j \notin \mathfrak{R}_2$. So the boundary condition (31) written for $k = j$ reads :

$$B_2 u_{0,j_{|x_2=0}} = 0.$$

Using the uniform Kreiss-Lopatinskii condition, it follows that $u_{0,j_{|x_2=0}} = 0$. So the amplitude $u_{0,j}$ satisfies the transport equation :

$$u_{0,j} = P_1^j u_{0,j} = P_2^j u_{0,j}, \quad \begin{cases} (\partial_t + v_j \cdot \nabla_x) Q_1^j u_{0,j} = 0, \\ u_{0,j_{|x_1=0}} = -S_1^j B_1 u_{0,n_{4|x_1=0}}, \\ u_{0,j_{|x_2=0}} = 0, \\ u_{0,j_{|t\le 0}} = 0. \end{cases}$$

To solve this transport equation, we use the flatness at the corner of $u_{0,n_{4|x_1=0}}$ to extend the problem in the half space $\{x_1 \ge 0\}$ by extending $u_{0,j}$ by zero to $\{x_2 < 0\}$, we integrate along the characteristics, then we restrict the constructed solution to the quarter space. The obtained solution $u_{0,j} \in H^K(\Omega_T)$, thanks to the fact $u_{0,n_4}{}_{|x_1=0}$ is flat at the corner.

One can also easily check that the obtained solution $u_{0,j}$ satisfies the property : let $\underline{x}_1 \geq 0$, $u_{0,j}{}_{|x_1=\underline{x}_1} \in H^K_f$. This extra regularity of $u_{0,j}$ will also be needed during the construction of higher order terms.

$\diamond n_4 \in \mathfrak{R}_1$

The determination of the amplitude associated with an evanescent index for the side $\partial\Omega_1$ (or even $\partial\Omega_2$), follows (in some sense) the same kind of ideas as the determination of amplitudes linked with oscillating indeces. Indeed, it will be easy to construct the amplitude linked with an evanescent index if we know its trace (on $\partial\Omega_1$ for elements of $\mathscr{I}_{ev1}$ and the trace on $\partial\Omega_2$ for indeces of $\mathscr{I}_{ev2}$).

However, we will in this proof treat the evanescent modes in only one block as in [9] ; that is why the associated amplitudes will not satisfy transport equations as in the oscillating case. Thus in a first time, we recall the evolution equations and the boundary conditions satisfied by such amplitudes and then we will give a method to solve these equations.

Plugging the ansatz (25) in the evolution equation of the corner problem (18) we have seen that the amplitude $U_{n,n_4,1}$ has to satisfy the cascade of equation :

$$\begin{cases} L_{n_4}(\partial_{X_1})U_{0,n_4,1} = 0, \\ L_{n_4}(\partial_{X_1})U_{n,n_4,1} + L(\partial)U_{n-1,n_4,1} = 0, \quad \forall n \geq 1, \end{cases} \tag{55}$$

where

$$L_{n_4}(\partial_{X_1}) := A_1\left(\partial_{X_1} - \mathscr{A}_1(\underline{\tau}, \xi_2^{n_4})\right).$$

The boundary conditions have also already been studied in the case $n_4 \in \mathfrak{R}_1$ and is given by equation (50). So $U_{0,n_4,1}$ has to satisfy the system :

$$\begin{cases} L_{n_4}(\partial_{X_1})U_{0,n_4,1} = 0, \\ B_1\left[\sum_{j\in\Phi(n_4)\cap(\mathscr{I}_{io}\cup\mathscr{I}_{ii})} u_{0,j} + U_{0,n_4,1}{}_{|X_1=0}\right]_{|x_1=0} = -B_1 u_{0,n_4}{}_{|x_1=0}, \\ U_{0,n_4,1}{}_{|t\leq 0} = 0, \end{cases} \tag{56}$$

Let us recall the following lemma from [9], which permits to solve (55) in the profile space $P_{ev,1}$.

Lemma 4.3 *For $i = 1, 2$, and $\underline{k} \in \mathfrak{R}_i$, let*

$$\mathbb{P}^{\underline{k}}_{ev,i}U(X_i) := e^{X_i\mathscr{A}_i(\underline{\tau},\xi^{\underline{k}}_{3-i})}P^{\underline{k}}_{s,i}U(0), \tag{57}$$

$$\mathbb{Q}^{\underline{k}}_{ev,i}F(X_i) := \int_0^{X_i} e^{(X_i-s)\mathscr{A}_i(\underline{\tau},\xi^{\underline{k}}_{3-i})}P^{\underline{k}}_{s,i}A_i^{-1}F(s)ds - \int_{X_i}^{+\infty} e^{(X_i-s)\mathscr{A}_i(\underline{\tau},\xi^{\underline{k}}_{3-i})}P^{\underline{k}}_{u,i}A_i^{-1}F(s)ds. \tag{58}$$

Then, for all $F \in P_{ev,i}$ the equation :

$$L_{\underline{k}}(\partial_{X_i})U = F,$$

admits a solution in $P_{ev,i}$. Moreover, this solution reads :

$$U = \mathbb{P}^{\underline{k}}_{ev,i}U + \mathbb{Q}^{\underline{k}}_{ev,i}F.$$

This lemma tells us that the evanescent amplitude of leading order $U_{0,n_4,1}$ satisfies $\mathbb{P}^{n_4}_{ev,1}U_{0,n_4,1} = U_{0,n_4,1}$. This relation is analogous of the polarization condition for oscillating phases and thanks to the definition of $\mathbb{P}^{n_4}_{ev,1}$, enables us to determine $U_{0,n_4,1}$ if we know its trace on $\{X_1 = 0\}$.

Unfortunately the system (56) does not give any information about this trace but only on the "double" trace on $\{x_1 = X_1 = 0\}$. This is determined by :

$$U_{0,n_4,1}{}_{|X_1=x_1=0} = -S^j_{s,1}B_1 u_{0,n_4}{}_{|x_1=0}.$$

It is then sufficient to lift the "double" trace in a "single" one. As in [9], let, for example, choose :

$$U_{0,n_4,1}(t,x,0) := -\chi(x_1) S^j_{s,1} B_1 u_{0,n_4|x_1=0},$$

where $\chi \in \mathscr{C}^\infty_c(\mathbb{R})$ satisfies $\chi(0) = 1$.

Now that the trace of $U_{0,n_4,1}$ on $\{X_1 = 0\}$ is determined, we can apply the operator $\mathbb{P}^{n_4}_{ev,1}$. Thus by construction, the amplitude

$$U_{0,n_4,1}(t,x,X_1) = -\chi(x_1) e^{X_1 \mathscr{A}_1(\underline{\tau},\xi_2^{n_4})} S^{n_4}_{s,1} B_1 u_{0,n_4|x_1=0}(t,x_2), \tag{59}$$

is a solution to the system of equations (56).

The determination of evanescent amplitudes for the side $\partial\Omega_2$ that can appear when we construct the amplitudes associated with indeces in $\Psi(n_3)$ or $\Psi(n_1)$, is totally similar. For example, for the indeces in $\Psi(n_3)$, we will start by determining $U_{0,n_3,2}$ on $\{x_2 = X_2 = 0\}$ by using the boundary condition, then we lift this "double" trace in a "single" one on $\{X_2 = 0\}$. This trace is finally propagated in the interior of Ω by the operator $\mathbb{P}^{n_3}_{ev,2}$ defined in (57).

Then we repeat this construction for all the indeces in the direct neighorhood of the loop, so the indeces whose amplitudes have still to be determined in partition (35) are :

$$\left(\cup_{a_l} A_{a_l} \setminus \{a_l\}\right) \bigcup \left(\cup_{b_m} B_{b_m} \setminus \{b_m\}\right) \bigcup \left(\cup_{c_q} C_{c_q} \setminus \{c_q\}\right) \bigcup \left(\cup_{d_r} D_{d_r} \setminus \{d_r\}\right),$$

that is to say, it only remains to determine the amplitudes linked with indeces in the trees of the partition (35). Before constructing these amplitudes, we will need to have a more precise description of the structure of those trees. It is the subject of the following paragraph.

4.2.4 Local structure of the trees.

Let us concentrate ourselves on the internal structure of the trees A_{a_l} appearing in the partition (35) of $\mathscr{I}$. The description for the "trees" B_{b_m}, C_{c_q} and D_{d_r} is, up to a few modifications, analogous and will not be given in details here. Let us recall that a tree $A_{a_{\underline{l}}}$ has for root an index $a_{\underline{l}} \in (\Psi(n_1) \cap \mathscr{I}_{oi}) \setminus \{n_4\}$ and is the set of indeces j linked with $a_{\underline{l}}$ by a sequence of type H (cf. Definition 4.3). To make the future notations more confortable we will denote $A_{a_{\underline{l}}} := A_{\underline{a}}$.

The following proposition has already been mentionned in paragraph 4.2.1, and is the main proposition needed to understand the structure of $A_{\underline{a}}$.

Proposition 4.6 *Let $j \in A_{\underline{a}}$, then there exists a unique sequence ℓ of type H linking j to $\underline{a}$.*

Proof : By contradiction, let $\ell = (\ell_1, \ell_2, ..., \ell_p)$ and $\ell' = (\ell'_1, \ell'_2, ..., \ell'_{p'})$, $\ell \neq \ell'$, be two sequences of type H which link j to $\underline{a}$. We will separate several cases depending on the oddness/evenness of the lengths p and p', and without loss of generality we assume that $p \leq p'$.

$\diamond$ $p, p' \in 2\mathbb{N}$.

We have to distinguish two differents subcases :

• If $\ell' = (\ell, \ell'_{p+1}, ..., \ell_{p'})$, then $\ell'_{p+1} \in \Phi(\ell'_{p'})$. This allows us to show that the sequence $(\ell'_{p+2}, ..., \ell_{p'})$ is a loop for the index ℓ'_{p+1} of length $p' - p - 1$. Thanks to Assumption 4.1, it is impossible.

• If $\ell' \neq (\ell, \ell'_{p+1}, ..., \ell'_{p'})$.

Let m be the first integer such that $\ell_m \neq \ell'_m$. From the preceding subcase, we can assume that $1 \leq m < p$. We will here deal with the case $m \in 2\mathbb{N}+1$ (the case $m \in 2\mathbb{N}$ can be treated in a similar way , up to modification of the type of sequence). We have, $\ell_m \in \Phi(\ell'_m)$.

We have, once again two different possibilities :

• There exists l, $m+1 < l \leq p$ such that $k_l = k'_l$. Then let $\underline{l}$ be the first integer l, $m+1 < l \leq p$ such that $k_l = k'_l$. Then if $\underline{l} \in 2\mathbb{N}$ (resp. $\underline{l} \in 2\mathbb{N}+1$), we have $\ell_{\underline{l}-1} \in \Psi(\ell'_{\underline{l}-1})$ (resp. $\ell_{\underline{l}-1} \in \Phi(\ell'_{\underline{l}-1})$). Consequently the sequence $(\ell'_m, ..., \ell'_{\underline{l}-1})$ is a sequence of type H linking $\ell_{\underline{l}-1}$ to ℓ_m, and the sequence

$(\ell_{m+1},...,\ell_{\underline{l}-2})$ is a sequence of type V linking $\ell_{\underline{l}-1}$ to ℓ_m. From this observation, we decude that the sequence $(\ell'_m,...,\ell'_{\underline{l}-1},\ell_{\underline{l}-1},\ell_{l-2},...,\ell_{m+1})$ is a loop for the index ℓ_m. Once again, this fact contradicts Assumption 4.1.

• If for all $q \in \{m+2,...,p\}$ the indeces ℓ_q and ℓ'_q are distinct, we can easily that $(\ell'_m,...,\ell'_p,\ell_p,...,\ell_{m-1})$ is a loop for ℓ_m.

We now consider the second subcase, that is to say :

$\diamond$ $p \in 2\mathbb{N}$, $p' \in 2\mathbb{N}+1$.

If $\ell' = (\ell,\ell'_{p+1},...,\ell'_{p'})$, we can show that $(\ell'_{p+1},...,\ell'_{p'})$ is a loop for j.

Whereas, if $\ell' \neq (\ell,\ell'_{p+1},...,\ell'_{p'})$, we can repeat the analysis made in the subcase $p,p' \in 2\mathbb{N}$ to treat the subcase, "There exists l, $m+1 < l \leq p$ such that $\ell_l = \ell'_l$". If for all $q \in \{m+2,...,p\}$ ℓ_q and ℓ'_q are distinct, we can easily show that $(\ell'_m,...,\ell'_p,j,\ell_p,...,\ell_{m-1})$ is a loop for ℓ_m.

The other case on the oddness of $p,p' \in 2\mathbb{N}+1$ is analogous, up to the inversion of the role played by the applications Φ and Ψ, to the case $p,p' \in 2\mathbb{N}$. This case, is left to the reader.

□

Remark As indicated in paragraph 4.2.1, the uniqueness of the sequence linking $\underline{j} \in A_{\underline{a}}$ to the root $\underline{a}$ depends, in a non trivial way, on Assumption 4.1.

Thanks to Proposition 4.6 it is now possible to give a more precise (and final) version of the properties satisfied by applications Φ and Ψ :

Proposition 4.7 *Let $\underline{j} \in A_{\underline{a}} \setminus \{\underline{a}\}$, we denote by $\ell = (\ell_1,...\ell_p)$ the sequence of type H linking $\underline{j}$ to $\underline{a}$. Then, according to the parity of p, we have :*

x') If $p \in 2\mathbb{N}$, then $\underline{j} \notin \mathscr{I}_{oi}$. Moreover, if $\underline{j} \in \mathscr{I}_{ev1} \cup \mathscr{I}_{ii}$ then $\Psi(\underline{j}) = \{\underline{j}\}$.

x'') If $p \in 2\mathbb{N}+1$, then $\underline{j} \notin \mathscr{I}_{io}$. Moreover, if $\underline{j} \in \mathscr{I}_{ev2} \cup \mathscr{I}_{ii}$ then $\Phi(\underline{j}) = \{\underline{j}\}$.

Proof : We will consider the case $p \in 2\mathbb{N}$. Let us first show that $\underline{j} \notin \mathscr{I}_{oi}$. By contradiction, we assume that $\underline{j} \in \mathscr{I}_{oi}$, but using the fact that the frequency set $\mathscr{F}$ is minimal we can assume that $\Psi(\underline{j}) \cap \mathscr{I}_{io} \neq \emptyset$.

Let $\underline{i} \in \Psi(\underline{j}) \cap \mathscr{I}_{io}$, according to the analysis made in paragraph 4.2.1, we have $\underline{i} \in A_{\underline{a}}$. Let $\ell' = (\ell'_1,...,\ell'_{p'})$ be the sequence linking $\underline{i}$ to the root $\underline{a}$. Reiterating the arguments used in the proof of Proposition 4.6, it is sufficient to study the case $\ell_i \neq \ell'_i$ for all i.

If $p' \in 2\mathbb{N}$, we can then show that $(\ell',i,j,\ell_p,...,\ell_2)$ is a loop with an odd number of elements for the index ℓ_1, whereas if $p' \in 2\mathbb{N}+1$, the sequence $(\ell',i,\ell_p,...\ell_2)$ is a loop for ℓ_1. Both cases contradict Assumption 4.1.

The proof of the assertion "If $j \in \mathscr{I}_{ii} \cup \mathscr{I}_{ev1}$, then $\Psi(j) = \{j\}$" follows the same reasoning.

□

The same proposition, up to some adaptations on the oddness/evenness according to the considered tree, is true for all trees in partition (35).

In terms of wave packet reflection, Proposition 4.7 states that, on one hand, during a reflection on the side $\partial\Omega_1$ (resp. $\partial\Omega_2$), an outgoing-incoming phase (resp. incoming-outgoing) can not generate (resp. incoming-outgoing) outgoing-incoming phases. This "natural" idea used in [17] is now rigorously justified.

4.2.5 Determination of the amplitudes for indeces in the trees.

Thanks to the precise description of the internal structure of the differents trees in the partition (35), it is easy to determine all the remaining amplitudes in the WKB expansion, and to conclude the construction of the leading order terms. Once again, we will here only deal with a tree $A_{\underline{a}}$. The construction is analogous for the other trees.

Let $\underline{j}$ be any index of $A_{\underline{a}}$, we will show that it is always possible to determine the amplitude $u_{0,\underline{j}}$. Thanks to Proposition 4.6, there exists a unique sequence of type H, denoted by $\ell^{\underline{j}} = \ell = (\ell_1,...,\ell_p)$, linking $\underline{j}$ to the root $\underline{a}$. The first step in the construction of the amplitude $u_{0,\underline{j}}$ is to remark that independently of the determination of $u_{0,\underline{j}}$, we can always first determine the amplitudes u_{0,ℓ_i}, $i = 1, ...p$.

Indeed, by definition of sequences of type H (see Definition 4.3), $\ell_1 \in \Phi(\underline{a}) \cap \mathscr{I}_{io}$. So, the amplitude u_{0,ℓ_1} satisfies a transport equation reading :

$$\left\{\begin{array}{l} (\partial_t + v_{\ell_1} \cdot \nabla_x) Q_1^{\ell_1} u_{0,\ell_1} = 0, \\ B_1 \left[\sum_{i\in\Phi(\ell_1)\cap(\mathscr{I}_{io}\cup\mathscr{I}_{ii})} u_{0,i_{|x_1=0}}\right] = -B_1 u_{0,\underline{a}_{|x_1=0}}, \quad \text{if } \ell_1 \in \mathscr{R}_1 \setminus \mathfrak{R}_1, \\ u_{0,\ell_{1|t\leq 0}} = 0, \end{array}\right. \tag{60}$$

$$\left\{\begin{array}{l} (\partial_t + v_{\ell_1} \cdot \nabla_x) Q_1^{\ell_1} u_{0,\ell_1} = 0, \\ B_1 \left[\sum_{i\in\Phi(\ell_1)\cap(\mathscr{I}_{io}\cup\mathscr{I}_{ii})} u_{0,i} + U_{0,\ell_1,1_{|X_1=0}}\right]_{|x_1=0} = -B_1 u_{0,\underline{a}_{|x_1=0}}, \quad \text{if } \ell_1 \in \mathfrak{R}_1, \\ u_{0,\ell_{1|t\leq 0}} = 0, \end{array}\right.$$

because all the elements of $\Phi(\ell_1)$ are linked with $\underline{a}$ by a sequence of length zero. Thanks to point x') of Proposition 4.7, it follows that $\Phi(\ell_1) \cap \mathscr{I}_{oi} = \{\underline{a}\}$. Consequenlty, multiplying (60) by $S_1^{\ell_1}$ (see Definition 4.6), we can write :

$$u_{0,\ell_{1|x_1=0}} = -S_1^{\ell_1} B_1 u_{0,\underline{a}_{|x_1=0}}.$$

This equation determines the trace of u_{0,ℓ_1} on the side $\partial\Omega_1$ because the amplitude $u_{0,\underline{a}}$ and its trace have already been determined in paragraph 4.2.3. Integrating (60) along the characteristics, we determine $u_{0,\ell_1} \in H^K(\Omega_T)$ and the trace $u_{0,\ell_{1|x_2=0}} \in H_f^K$.

Then we are interested in the construction of the amplitude u_{0,ℓ_2}, by definition of type H sequences, $\ell_2 \in \Psi(\ell_1) \cap \mathscr{I}_{oi}$. Once again, we can apply Proposition 4.7 to show that $\Psi(\ell_2) \cap \mathscr{I}_{io} = \{\ell_1\}$. This allows us to rewrite the transport equation on u_{0,ℓ_2} under the form :

$$\left\{\begin{array}{l} (\partial_t + v_{\ell_2} \cdot \nabla_x) Q_2^{\ell_2} u_{0,\ell_2} = 0, \\ u_{0,\ell_{2|x_2=0}} = -S_2^{\ell_2} B_2 u_{0,\ell_{1|x_2=0}}, \\ u_{0,\ell_{2|t\leq 0}} = 0, \end{array}\right. \tag{61}$$

and we solve this equation by integration along the characteristics.

We can reiterate the same kind of resolutions of transport equations for all the indeces of the sequence ℓ. This operation permits us to construct all the amplitudes u_{0,ℓ_l}, $l = 1, ..., p$. The important point in these recursive resolutions is that since the first amplitude $u_{0,\underline{a}}$ has its trace on $\partial\Omega_1$ in H_f^K, this flatness at the corner is transmitted to all the amplitudes indexed by the sequence ℓ.

Indeed, integration along the characteristics gives an explicit formula and it is easy to see on this formula that the traces of the u_{0,ℓ_l} are in H_f^K, for all $1 \leq l \leq p$.

Once we have constructed all the amplitudes associated by the indeces of ℓ, it is easy to determine the amplitude $u_{0,\underline{j}}$. We distinguish the following cases depending of the nature of the index $\underline{j}$.

$\diamond$ $\underline{j} \in \mathscr{I}_{io}$ (resp. $\underline{j} \in \mathscr{I}_{oi}$).

Proposition 4.7 tells us that an index in $\mathscr{I}_{io} \cap A_{\underline{a}}$ (resp. $\mathscr{I}_{oi}$) can appear only after an even (resp. odd) number of reflections, in other terms the lenght of the sequence ℓ, $p \in 2\mathbb{N}$ (resp. $p \in 2\mathbb{N}+1$). Using the fact that $\underline{j} \in \mathscr{I}_{io}$ (resp. $\underline{j} \in \mathscr{I}_{io}$), to construct the amplitude $u_{0,\underline{j}}$ it is sufficient to know $u_{0,\underline{j}_{|x_1=0}}$ (resp. $u_{0,\underline{j}_{|x_2=0}}$). But, Proposition 4.7 implies that ℓ_p is the only element of $\mathscr{I}_{oi}$ (resp. $\mathscr{I}_{io}$) in $\Phi(\underline{j})$ (resp. $\Psi(\underline{j})$), multiplying by $S_1^{\underline{j}}$ (resp. $S_2^{\underline{j}}$), we can determine the trace $u_{0,\underline{j}_{|x_1=0}}$ (resp. $u_{0,\underline{j}_{|x_2=0}}$) as a function of the trace $u_{0,\ell_{p|x_1=0}}$ (resp. $u_{0,\ell_{p|x_2=0}}$). Consequently the amplitude $u_{0,\underline{j}}$ is constructed.

Moreover, we can show that $u_{0,\underline{j}} \in H^K(\Omega)$, and that its traces on the sides $\partial\Omega_1$ and $\partial\Omega_2$ are in H_f^K.

$\diamond$ $\underline{j} \in \mathscr{I}_{ii}$.

An incoming-incoming index may appear after an even number of reflections as well as after an odd number of reflections. We will here deal with the case $p \in 2\mathbb{N}$, the case $p \in 2\mathbb{N}+1$ is totally similar. Proposition 4.7

implies, on the one hand that ℓ_p is the only index in $\Phi(\underline{j}) \cap \mathscr{I}_{oi}$ and on the other hand that $\Psi(\underline{j}) = \{\underline{j}\}$. So the boundary conditions for the amplitude $u_{0,\underline{j}}$ can be writen under the form :

$$\begin{aligned} u_{0,\underline{j}_{|x_1=0}} &= -S_1^{\underline{j}} B_1 u_{0,\ell_{p|x_1}}, \\ u_{0,\underline{j}_{|x_2=0}} &= 0. \end{aligned}$$

It follows that the amplitude $u_{0,\underline{j}}$ satisfies the incoming-incoming transport equation :

$$\begin{cases} (\partial_t + v_{\underline{j}} \cdot \nabla_x) Q_1^{\underline{j}} u_{0,\underline{j}} = 0, \\ u_{0,\underline{j}_{|x_1=0}} = -S_1^{\underline{j}} B_1 u_{0,\ell_{p|x_1=0}}, \\ u_{0,\underline{j}_{|x_2=0}} = 0, \\ u_{0,\underline{j}_{|t\leq 0}} = 0. \end{cases} \tag{62}$$

To solve this equation, we extend the source term $-S_1^{\underline{j}} B_1 u_{0,\ell_{p|x_1=0}}$ by zero on $\{x_2 < 0\}$ (this extension gives a regular function because $u_{0,\ell_{p|x_1=0}} \in H_f^K$) then we restrict to $\{x_2 \geq 0\}$ the solution to the transport equation in the half space $\{x_1 \geq 0, x_2 \in \mathbb{R}\}$.

Consequently we have constructed $u_{0,\underline{j}} \in H^K(\Omega_T)$, such that for all $\underline{x}_1 \geq 0$, $u_{0,\underline{j}_{|x_1=\underline{x}_1}} \in H_f^K$.

$\diamond$ $\underline{j} \in \mathfrak{R}_1$ (resp. $\underline{j} \in \mathfrak{R}_2$).
As in the case $\underline{j} \in \mathscr{I}_{io}$ (resp. $\underline{j} \in \mathscr{I}_{oi}$), Proposition 4.7 tells us that an evanescent index for the side $\partial\Omega_1$ (resp. $\partial\Omega_2$) can only appear after an even (resp. odd) number of reflections. Moreover, Proposition 4.7 also implies that the only index of $\mathscr{I}_{oi}$ (resp. $\mathscr{I}_{io}$) and $\Phi(\underline{j})$ (resp. $\Psi(\underline{j})$) is ℓ_p. We are now interested in the construction of the evanescent amplitude $U_{0,\underline{j},1}$ (resp. $U_{0,\underline{j},2}$).

Repeating the construction described in paragraph 4.2.3, to determine the amplitude $U_{0,\underline{j},1}$ (resp. $U_{0,\underline{j},2}$), it is sufficient to start by determining the "double trace" on $\{X_1 = x_1 = 0\}$ (resp. $\{X_2 = x_2 = 0\}$). Using the fact that ℓ_p is the only element of $\mathscr{I}_{oi}$ (resp. $\mathscr{I}_{io}$) in $\Phi(\underline{j})$ (resp. $\Psi(\underline{j})$), allows us to show that this "double trace" is given by :

$$U_{0,\underline{j},1}(t,0,x',0) = -S_{s,1}^{\underline{j}} B_1 u_{0,\ell_{p|x_1=0}}, \quad \left(\text{resp. } U_{0,\underline{j},2}(t,x',0,0) = -S_{s,2}^{\underline{j}} B_2 u_{0,\ell_{p|x_2=0}}\right).$$

We then lift this double trace in a "single" one by setting :

$$U_{0,\underline{j},1}(t,x,0) := -\chi(x_1) S_{s,1}^{\underline{j}} B_1 u_{0,\ell_{p|x_1=0}}, \quad \left(\text{resp. } U_{0,\underline{j},2}(t,x,0) := -\chi(x_2) S_{s,2}^{\underline{j}} B_2 u_{0,\ell_{p|x_2=0}}\right),$$

where $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ is such that $\chi(0) = 1$. Finally, lemma 4.3 shows that the function $U_{0,\underline{j},1}$ (resp. $U_{0,\underline{j},2}$) defined by :

$$\begin{aligned} U_{0,\underline{j},1}(t,x,X_1) &= -\chi(x_1) e^{X_1 \mathscr{A}_1} S_{s,1}^{\underline{j}} B_1 u_{0,\ell_{p|x_1=0}}, \\ \Big(\text{resp. } U_{0,\underline{j},2}(t,x,X_2) &= -\chi(x_2) e^{X_2 \mathscr{A}_2} S_{s,2}^{\underline{j}} B_2 u_{0,\ell_{p|x_2=0}}\Big), \end{aligned} \tag{63}$$

is a solution to the cascades of equations (26)-(31) and (32) written for $n = 0$ and $k = \underline{j}$.

In this paragraph we have shown that an arbitrary amplitude in the tree $A_{\underline{a}}$ could always be constructed. As a consequence, all the amplitudes in the tree $A_{\underline{a}}$ can be determined. Then it is sufficient to repeat the method of construction for each tree in the partition (35). So we have constructed all the amplitudes associated with indeces in $\mathscr{I} \setminus \{n_1, n_2, n_3, n_4\}$, and we have thus finished the construction of the leading term in the WKB expansion. We summarize the analysis with the following proposition :

Proposition 4.8 *Under Assumptions 2.1-2.2 on the complete for reflexions corner problem* (18)*, under Assumptions 4.1 and 4.3, there exist functions* $(u_{0,i})_{i\in\mathscr{I}_{os}}$*,* $(U_{0,i,1})_{i\in\mathfrak{R}_1}$ *and* $(U_{0,i,2})_{i\in\mathfrak{R}_2}$ *satisfying the cascades* *of equations* (26)-(31) *and* (32) *written for* $n = 0$.

Moreover, the functions $(u_{0,i})_{i\in\mathscr{I}_{os}}$ *admit the following regularity : for all* $T > 0$,
◇ *if* $i \in \mathscr{I}_{io} \cup \mathscr{I}_{oi}$, $u_{0,i} \in H^K(\Omega_T)$ *and the traces* $u_{0,i_{|x_1=0}}$ *and* $u_{0,i_{|x_2=0}}$ *are in* H^K_f.
◇ *If* $i \in \mathscr{I}_{ii}$ *then* $u_{0,i} \in H^K(\Omega_T)$. *Moreover, if* $\Psi(i) = \{i\}$ *(resp.* $\Phi(i) = \{i\}$*) then for all* $\underline{x}_1 > 0$ *(resp.* $\underline{x}_2 > 0$*), the trace* $u_{0,i_{|x_1=\underline{x}_1}}$ *(resp.* $u_{0,i_{|x_2=\underline{x}_2}}$*) is* H^K_f.

The functions $(U_{0,i,1})_{i\in\mathfrak{R}_1}$ *(resp.* $(U_{0,i,2})_{i\in\mathfrak{R}_2}$*) are in* P_{ev1} *(resp.* P_{ev2}*).*

4.2.6 Construction of the higher order terms in the WKB expansion.

The construction for the higher order terms in the WKB expansion looks like the construction of the leading order term. In particular, the order of resolution will be the same : we start with the amplitudes on the loop, then we show that the knowledge of these amplitudes is sufficient to construct any amplitude in the trees of the partition (35). In this paragraph we only give the main steps of the construction of the term of order ε, without all details. Let us begin with the oscillating amplitudes.

For $\underline{k} \in \mathscr{I}_{os}$, the amplitude $u_{1,\underline{k}}$ satisfies the equations :

$$\begin{cases} i\mathscr{L}(d\varphi_k)u_{1,\underline{k}} + L(\partial)u_{0,\underline{k}} = 0, \\ u_{1,\underline{k}_{|t\le 0}} = 0, \end{cases} \tag{64}$$

with the two boundary conditions :

$$\begin{aligned} B_1\left[\sum_{k\in\Phi(\underline{k})\cap\mathscr{I}_{os}} u_{1,k}\right]_{|x_1=0} &= 0, \text{ if } \underline{k} \notin \mathfrak{R}_1, \\ B_1\left[\sum_{k\in\Phi(\underline{k})\cap\mathscr{I}_{os}} u_{1,k} + U_{1,\underline{k},1_{|X_1=0}}\right]_{|x_1=0} &= 0, \text{ if } \underline{k} \in \mathfrak{R}_1, \end{aligned} \tag{65}$$

and

$$\begin{aligned} B_2\left[\sum_{k\in\Psi(\underline{k})\cap\mathscr{I}_{os}} u_{1,k}\right]_{|x_2=0} &= 0, \text{ if } \underline{k} \notin \mathfrak{R}_2 \\ B_2\left[\sum_{k\in\Psi(\underline{k})\cap\mathscr{I}_{os}} u_{1,k} + U_{1,\underline{k},2_{|X_2=0}}\right]_{|x_2=0} &= 0, \text{ if } \underline{k} \in \mathfrak{R}_2. \end{aligned} \tag{66}$$

In a classical way, we compose the first equation of (64) by the partial inverse $R^{\underline{k}}_1$ if $\underline{k} \in \mathscr{I}_{io}$, $R^{\underline{k}}_2$ if $\underline{k} \in \mathscr{I}_{oi}$ and by $R^{\underline{k}}_1$ or $R^{\underline{k}}_2$ if $\underline{k} \in \mathscr{I}_{ii}$. Let us recall that this partial inverse satisfies : for $i = 1, 2$,

$$R^{\underline{k}}_i\mathscr{L}(d\varphi_{\underline{k}}) = I - P^{\underline{k}}_i, \quad P^{\underline{k}}_i R^{\underline{k}}_i = R^{\underline{k}}_i Q^{\underline{k}}_i = 0.$$

The first equation of (64), after this composition, reads :

$$\left(I - P^{\underline{k}}_i\right) u_{1,\underline{k}} = iR^{\underline{k}}_i L(\partial)u_{0,\underline{k}}, \tag{67}$$

and determines in a unique way the unpolarized part of $u_{1,\underline{k}}$. Indeed, at this stage of the analysis the term of the right hand side of (67) has already been constructed. As $u_{0,\underline{k}} \in H^K(\Omega_T)$ and its traces on the sides $\partial\Omega_1$ and $\partial\Omega_2$ are in H^K_f, the unpolarized part of $u_{1,\underline{k}}$ belongs to $H^{K-1}(\Omega_T)$ with traces in H^{K-1}_f. To complete the construction of the oscillating amplitude $u_{1,\underline{k}}$, we just have to construct its polarized part ; that is to say $P^{\underline{k}}_1 u_{1,\underline{k}}$ (or equivalently $P^{\underline{k}}_2 u_{1,\underline{k}}$).

To determine the polarized part, we will repeat with some modifications, the method described for the leading order. First, we remark that the evolution equation for the amplitude $u_{2,\underline{k}}$ is :

$$i\mathscr{L}(d\varphi_{\underline{k}})u_{2,\underline{k}} + L(\partial)u_{1,\underline{k}} = 0,$$

and reads, after composition by $Q_1^{\underline{k}}$ for $\underline{k} \in \mathscr{I}_{io}$, by $Q_2^{\underline{k}}$ for $\underline{k} \in \mathscr{I}_{oi}$ and by $Q_1^{\underline{k}}$ or $Q_2^{\underline{k}}$ for $\underline{k} \in \mathscr{I}_{ii}$:

$$Q_i^{\underline{k}} L(\partial) P_i^{\underline{k}} u_{1,\underline{k}} = -Q_i^{\underline{k}} L(\partial)(I - P_i^{\underline{k}}) u_{1,\underline{k}} = -iQ_i^{\underline{k}} L(\partial) R_i^{\underline{k}} L(\partial) u_{0,\underline{k}}.$$

Thanks to Lax lemma [8], this equation is a transport equation with speed $v_{\underline{k}}$ on the polarized part $P_i^{\underline{k}} u_{1,\underline{k}}$. As a consequence, $Q_i^{\underline{k}} P_i^{\underline{k}} u_{1,\underline{k}}$ satisfies the same transport equation (with a non-zero source term in the interior of Ω) as the transport equation satisfied by $u_{0,\underline{k}}$. This observation leads us to apply the same method of construction as in paragraphs 4.2.3 and 4.2.5.

More precisely, we start with the indeces on the loop, to fix the ideas, we will describe the construction of u_{1,n_4}. We have already seen that its unpolarized part is known. To construct the polarized part of u_{1,n_4}, since it travels with an outgoing-incoming velocity, we need to know its trace on $\partial\Omega_2$. Repeating the computation made in paragraph 4.2.2 we obtain an invertiblity condition which reads :

$$(I - \mathbb{T}) P_1^{n_1} u_{1,n_1}{}_{|x_2=0} = G_1,$$

where $G_1 \in H_f^{K-1}$ only depends on the unpolarized traces of the amplitudes associated with the elements of the loop. Assumption 4.2 implies that $P^{n_4} u_{1,n_4}$ is solution to the transport equation :

$$\begin{cases} (\partial_t + v_{n_4} \cdot \nabla_x) Q_2^{n_4} P_2^{n_4} u_{1,n_4} = -iQ_2^{n_4} L(\partial) R_2^{n_4} L(\partial) u_{0,n_4}, \\ P_2^{n_4} u_{1,n_4}{}_{|x_2=0} = -S_2^{n_4} B_2 \left[(I - \mathbb{T})^{-1} G_1 + (I - P_2^{n_4}) u_{1,n_4}{}_{|x_2=0} + (I - P_1^{n_1}) u_{1,n_1}{}_{|x_2=0} \right], \\ P_2^{n_4} u_{1,n_4}{}_{|t\leq 0} = 0. \end{cases}$$

All the source terms in this equation are known, so we can integrate along the characteristics to determine $P_2^{n_4} u_{1,n_4}$. The source term in the interior is $H^{K-2}(\Omega_T)$ and the source term on the boundary is H_f^{K-1}, so the solution $P_2^{n_4} u_{1,n_4} \in H^{K-2}(\Omega_T)$ with traces on $\partial\Omega_1$ and $\partial\Omega_2$ in H_f^{K-2}. The fact that the construction of the term of order one in ε needs two derivatives is classical, and more generally, the construction of the term of order N_0 needs $2N_0$ derivatives on the $u_{0,j}$'s.

When the amplitudes associated with indeces on the loop are determined, the construction of the polarized parts of the other oscillating amplitudes follows exactly the same method. In particular the "order" of resolution is the same order as the "order" described in paragraph 4.2.5. That is why we will not give more details about this construction.

We are now interested in the construction of the evanescent amplitudes of order ε. Although these amplitudes do not satisfy transport equations, the method of construction is based on the same ideas as the method for oscillating amplitudes. Indeed, we remark that the amplitudes $U_{1,k,i}$ can be decomposed in a polarized part (whose construction will use the technics of the construction of $U_{0,k,i}$) and an unpolarized part only depending of the known amplitude $U_{0,k,i}$.

In this paragraph we will only consider evanescent amplitudes for the side $\partial\Omega_1$, so let $\underline{k} \in \mathfrak{R}_1$. The amplitude $U_{1,\underline{k},1}$ satisfies the system of equations :

$$\begin{cases} L_{\underline{k}}(\partial_{X_1}) U_{1,\underline{k},1} + L(\partial) U_{0,\underline{k},1} = 0, \\ B_1 \left[\sum_{k \in \Phi(\underline{k}) \cap \mathscr{I}_{os}} u_{1,k} + U_{1,\underline{k},1}{}_{|X_1=0} \right]_{|x_1=0} = 0, \\ U_{1,\underline{k},1}{}_{|t\leq 0} = 0, \end{cases}$$

but thanks to lemma 4.3, we know that the first equation of this system has a solution reading :

$$U_{1,\underline{k},1} = \mathbb{P}_{ev1}^{\underline{k}} U_{1,\underline{k},1} + \mathbb{Q}_{ev1}^{\underline{k}} L(\partial) U_{0,\underline{k},1},$$

where we recall that the projectors $\mathbb{P}^{\underline{k}}_{ev1}$ and $\mathbb{Q}^{\underline{k}}_{ev1}$ are defined in (57) and (58). Using the fact that the amplitude $U_{0,\underline{k},1}$ has already been constructed, the unpolarized part of $U_{1,\underline{k},1}$, namely $\mathbb{Q}^{\underline{k}}_{ev1}L(\partial)U_{0,\underline{k},1}$, is known. It is thus sufficient to construct the polarized part of $U_{1,\underline{k},1}$, namely $\mathbb{P}^{\underline{k}}_{ev1}U_{1,\underline{k},1}$. To do that, we repeat the construction used for $U_{0,k,1}$. By definition of $\mathbb{P}^{\underline{k}}_{ev1}$, $\mathbb{P}^{\underline{k}}_{ev1}U_{1,\underline{k},1}$ will be determined if we can construct the trace of $U_{1,\underline{k},1}$ on $\{X_1 = 0\}$.

Firstly, the boundary condition (68) and Proposition 4.7 give the "double trace" on $\{x_1 = X_1 = 0\}$. More precisely, this "double trace" is given by :

$$U_{1,\underline{k},1_{|X_1=x_1=0}} = -S^{\underline{k}}_{s,1}B_1u_{1,\underline{k}_{|x_1=0}},$$

where the source term is known because we have already constructed the oscillating amplitudes of order ε. To conclude we lift this "double trace" on $\{x_1 = X_1 = 0\}$ in a "single" trace $\{X_1 = 0\}$ exactly as it has been done for $U_{0,\underline{k},1}$, and then we apply the operator $\mathbb{P}^{\underline{k}}_{ev1}$.

So the construction of the WKB expansion for the corner problem (18) is complete. To summarize we give the following Theorem :

Theorem 4.1 *Under Assumptions 2.1-2.2 on the complete for the reflections corner problem* (18)*, under Assumptions 4.1 and 4.3, if* $[\cdot]$ *denotes the integer part function, then there exist functions* $(u_{n,k})_{n\leq[\frac{K}{2}],\,k\in\mathscr{I}_{os}}$*,* $(U_{n,k,1})_{\leq[\frac{K}{2}],\,k\in\mathfrak{R}_1}$ *and* $(U_{n,k,2})_{\leq[\frac{K}{2}],\,k\in\mathfrak{R}_2}$ *satisfying the cascades of equations* (26)*-*(31) *and* (32)*.*

Moreover, the functions $u_{n,k}$ *admit the following regularity : for all* $T > 0$*,*
$\diamond$ *if* $k \in \mathscr{I}_{io} \cup \mathscr{I}_{oi}$ *then* $u_{n,k} \in H^{K-2n}(\Omega_T)$ *and the traces* $u_{n,k_{|x_1=0}}$ *and* $u_{n,k_{|x_2=0}}$ *are in* H^{K-2n}_f*.*
$\diamond$ *If* $n \in \mathscr{I}_{ii}$*, then* $u_{n,k} \in H^{K-2n}(\Omega_T)$*. Moreover, if* $\Psi(k) = \{k\}$ *(resp.* $\Phi(k) = \{k\}$*) then for all* $\underline{x}_1 > 0$ *(resp.* $\underline{x}_2 > 0$*), the trace* $u_{n,k_{|x_1=\underline{x}_1}}$ *(resp.* $u_{n,k_{|x_2=\underline{x}_2}}$*) is* H^{K-2n}_f*.*

The $U_{n,k,1}$ *(resp.* $U_{n,k,2}$*) are in* P_{ev1} *(resp.* P_{ev2}*).*

4.3 Justification of the WKB expansion.

In this paragraph we show that, if the corner problem (18) is strongly well-posed, the truncated WKB expansion constructed in the preceding paragraph is a good approximation to the exact solution u^ε of the corner problem (18). Let us recall what we mean strong well-posedness :

Definition 4.8 *The corner problem is said to be strongly well-posed if for all* $f \in L^2(\Omega_T)$*,* $g_1 \in L^2(\partial\Omega_{1,T})$ *and* $g_2 \in L^2(\partial\Omega_{2,T})$ *zero for negative times, the system :*

$$\begin{cases} \partial_t u + A_1\partial_1 u + A_2\partial_2 u = f, \\ B_1 u_{|x_1=0} = g_1, \\ B_2 u_{|x_2=0} = g_2, \\ u_{|t\leq 0} = 0, \end{cases}$$

admits a unique solution $u \in L^2(\Omega_T)$*, with traces in* $L^2(\partial\Omega_{1,T})$ *and* $L^2(\partial\Omega_{2,T})$*, satisfying the energy estimate :*

$$\|u\|^2_{L^2(\Omega_T)} + \|u_{|x_1=0}\|^2_{L^2(\partial\Omega_{1,T})} + \|u_{|x_2=0}\|^2_{L^2(\partial\Omega_{2,T})} \leq C_T\left(\|f\|^2_{L^2(\Omega_T)} + \|g_1\|^2_{L^2(\partial\Omega_{1,T})} + \|g_2\|^2_{L^2(\partial\Omega_{2,T})}\right). \quad (68)$$

Let us recall that the strong well-posedness of the corner problem is demonstrated for the particular class of symmetric corner problems with strictly dissipative boundary conditions.

To justify the convergence of the WKB expansion, we need to be sure that the amplitudes are regular enough. The regularity has already been studied for the oscillating amplitudes. Concerning the evanescents amplitudes, the following Proposition shows that they are regular and also gives their size according to the small parameter ε.

Proposition 4.9 *Let U be an element of $P_{ev,1}$ (resp. $P_{ev,2}$). Then the functions $U\left(t,x,\frac{x_1}{\varepsilon}\right)$ and $(L(\partial)U(t,x,X_1))_{|X_1=\frac{x_1}{\varepsilon}}$ (resp. $U\left(t,x,\frac{x_2}{\varepsilon}\right)$ and $(L(\partial)U(t,x,X_2))_{|X_2=\frac{x_2}{\varepsilon}}$) are $O(\varepsilon^{\frac{1}{2}})$ in $L^2(\Omega_T)$.*

We refer to [1] for a proof of this result.

Before showing that the truncated WKB expansion is a good approximation to the exact solution to the corner problem (18), we have to make sure that the truncated WKB expansion makes sense. Indeed, we have seen in paragraph 3.4 that when there was an infinite number of phases generated by successive reflections, it was not clear that the sum of all amplitudes defining the WKB expansion converges. That is why, to avoid this difficulty we will restrict ourselves to a finite number of phases :

Assumption 4.4 *We assume that the number of phases generated by successive reflections on the sides $\partial\Omega_1$ and $\partial\Omega_2$ is finite. That is to say, $\#\mathscr{F} < +\infty$.*

With this extra assumption, it is clear that the truncated WKB expansion makes sense. The main Theorem of this article is :

Theorem 4.2 *Under Assumptions 2.1-2.2 for the complete for reflexion corner problem* (18) *and under Assumptions 4.1 and 4.4, 4.3, then*

For $N_0 \in \mathbb{N}$, with $N_0 \leq \left[\frac{K}{2} - \frac{3}{2}\right]$, we denote by u^ε_{app,N_0} the geometric optics expansion truncated at order N_0 defined by :

$$\begin{aligned} u^\varepsilon_{app,N_0} &:= \sum_{k\in\mathscr{I}_{os}} e^{\frac{i}{\varepsilon}\varphi_k(t,x)} \sum_{n=0}^{N_0} \varepsilon^n u_{n,k}(t,x) \\ &+ \sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}(t,x_2)} \sum_{n=0}^{N_0} \varepsilon^n U_{n,k,1}\left(t,x,\frac{x_1}{\varepsilon}\right) + \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}(t,x_1)} \sum_{n=0}^{N_0} \varepsilon^n U_{n,k,2}\left(t,x,\frac{x_2}{\varepsilon}\right), \end{aligned}$$

where functions $u_{n,k}$, $U_{n,k,1}$ and $U_{n,k,2}$ are given by Theorem 4.8. Then, if the corner problem (18) *is strongly well-posed, let u^ε be its exact solution, the error $u^\varepsilon - u^\varepsilon_{app,N_0}$ is $O(\varepsilon^{N_0+1})$ in $L^2(\Omega_T)$.*

Proof : Since we assumed that $N_0 \leq \left[\frac{K}{2} - \frac{3}{2}\right]$, the term of order ε^{N_0+1} of the WKB expansion makes sense and is at least in $H^1(\Omega_T)$. By construction of the $u_{n,k}$, $U_{n,k,1}$ and $U_{n,k,2}$, for $n \leq N_0+1$, the remainder $u^\varepsilon - u^\varepsilon_{app,N_0+1}$ satisfies the corner problem :

$$\begin{cases} L(\partial)(u^\varepsilon - u^\varepsilon_{app,N_0+1}) = f^\varepsilon_{N_0+1}, \\ B_1(u^\varepsilon - u^\varepsilon_{app,N_0+1})_{|x_1=0} = 0, \\ B_2(u^\varepsilon - u^\varepsilon_{app,N_0+1})_{|x_2=0} = 0, \\ (u^\varepsilon - u^\varepsilon_{app,N_0+1})_{|t\leq 0} = 0. \end{cases} \tag{69}$$

with

$$\begin{aligned} f^\varepsilon_{N_0+1} := \varepsilon^{N_0+1}\Bigg[\sum_{k\in\mathscr{I}_{os}} & e^{\frac{i}{\varepsilon}\varphi_k} \quad L(\partial)u_{N_0+1,k} + \sum_{k\in\mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}} \left(L(\partial)U_{N_0+1,k,1}\right)_{|X_1=\frac{x_1}{\varepsilon}} \\ &+ \sum_{k\in\mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}} \left(L(\partial)U_{N_0+1,k,2}\right)_{|X_2=\frac{x_2}{\varepsilon}}\Bigg]. \end{aligned}$$

But the corner problem (18) is supposed to be strongly well-posed, so we can use the energy estimate (68), to obtain :

$$\|u^\varepsilon - u^\varepsilon_{app,N_0+1}\|_{L^2(\Omega_T)} \leq C_T \|f^\varepsilon_{N_0+1}\|_{L^2(\Omega_T)}.$$

The right hand side can be estimated by :

$$\begin{aligned}
\|f^{\varepsilon}_{N_0+1}\|_{L^2(\Omega_T)} &\leq \varepsilon^{N_0+1}\left[\sum_{k\in\mathscr{I}_{os}}\|L(\partial)u_{N_0+1,k}\|_{L^2(\Omega_T)}+\sum_{k\in\mathscr{I}_{ev1}}\|L(\partial)U_{N_0+1,k,1}\left(\cdot,\cdot,X_1\right)_{|X_1=\frac{x_1}{\varepsilon}}\|_{L^2(\Omega_T)}\right.\\
&+\left.\sum_{k\in\mathscr{I}_{ev2}}\|L(\partial)U_{N_0+1,k,2}\left(\cdot,\cdot,X_2\right)_{|X_2=\frac{x_2}{\varepsilon}}\|_{L^2(\Omega_T)}\right],\\
&\leq C\varepsilon^{N_0+1},
\end{aligned}$$

because, according to Proposition 4.9, $(L(\partial)U_{N_0,k,1})_{|X_1=\frac{x_1}{\varepsilon}}$ and $(L(\partial)U_{N_0+1,k,2})_{|X_2=\frac{x_2}{\varepsilon}}$ are $O(\varepsilon^{\frac{1}{2}})$ in $L^2(\Omega_T)$ whereas $L(\partial)u_{N_0+1,k}$ are $O(1)$ in $L^2(\Omega_T)$, because $u_{N_0+1,k}$ is at least in $H^1(\Omega_T)$.

We thus have shown that :

$$\|u^{\varepsilon}-u^{\varepsilon}_{app,N_0+1}\|_{L^2(\Omega_T)}\leq C_T\varepsilon^{N_0+1},$$

and we conclude by triangle inequality.

□

4.4 Study of the invertibility condition (46).

In this paragraph we will give a sufficient (and also necessary in several relevant cases) condition ensuring that the invertibility condition (46) is satisfied. Let us recall that this condition reads :

$$u_{0,n_{1|x_2=0}}(t,x_1)-Su_{0,n_{1|x_2=0}}(t-\alpha x_1,\beta x_1)=S_1^{n_1}g(t+\delta x_1,\kappa x_1), \tag{70}$$

with $\alpha,\beta>0$, and $\delta<0$, $\kappa>0$. The exact expressions of this parameters are given by :

$$\begin{aligned}
S &:= S_1^{n_1}B_1S_2^{n_2}B_2S_1^{n_3}B_1S_2^{n_4}B_2,\\
\alpha &:= -\frac{1}{v_{n_1,1}}\left[-1+\frac{v_{n_1,2}}{v_{n_2,2}}-\frac{v_{n_1,2}v_{n_2,1}}{v_{n_2,2}v_{n_3,1}}+\frac{v_{n_1,2}v_{n_2,1}v_{n_3,2}}{v_{n_2,2}v_{n_3,1}v_{n_4,2}}\right],\\
\beta &:= \frac{v_{n_4,1}}{v_{n_4,2}}\frac{v_{n_3,2}}{v_{n_3,1}}\frac{v_{n_2,1}}{v_{n_2,2}}\frac{v_{n_1,2}}{v_{n_1,1}}.
\end{aligned}$$

If we assume that $\dim\ker\mathscr{L}(i\underline{\tau},\xi_1^{n_1},\xi_2^{n_1})=1$ (Assumption which is automatically satisfy in the stricly hyperbolic framework), then (70) is in fact a scalar equation :

$$u(t,x_1)-\widetilde{S}u(t-\alpha x_1,\beta x_1)=G(t,x_1), \tag{71}$$

where thanks to the polarization condition we write $u_{0,n_{1|x_2=0}}(t,x_1)=u(t,x_1)e_{n_1}$, with e_{n_1} chosen such that $\ker(\mathscr{L}(i\underline{\tau},\xi_1^{n_1},\xi_2^{n_1}))=\text{Span }e_{n_1}$. The scalar $\widetilde{S}$ is defined by the equality

$$Se_{n_1}=\widetilde{S}e_{n_1},$$

and without loss of generality we can assume that $\widetilde{S}\neq 0$.
In all what follows

It will be more convenient to rewrite (71) under the following form :

$$(I-\mathbb{T})u=G, \tag{72}$$

where :

$$(\mathbb{T}u)(t,x_1) := \widetilde{S}u(t-\alpha x_1,\beta x_1). \tag{73}$$

A sufficient condition for (70) (and thus also (71)) to have a unique solution in the profiles space $L^2(]-\infty,T]\times\mathbb{R}_+)$ is given by the following Theorem :

Theorem 4.3 *If*
$$|S| < \sqrt{\beta},$$
then for all $\gamma > 0$*, for all* $G \in L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$*, the functional equation* (70) *admits a unique solution* $u \in L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$*, polarized on* $\ker \mathscr{L}(i\underline{\tau}, \xi_1^{n_1}, \xi_2^{n_1})$ *satisfying :*
$$\|u\|_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)} \leq C\|G\|_{L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)},$$
where C *does not depend on the parameter* γ*.*

In particular, for all $T > 0$*, if* $G \in L^2(\partial\Omega_{2,T})$ *and is zero for negative times, then* (70) *has a unique solution* $u \in L^2(\partial\Omega_{2,T})$*, polarized on* $\ker(\mathscr{L}(i\underline{\tau}, \xi_1^{n_1}, \xi_2^{n_1})$*, and satisfying :*
$$\|u\|_{L^2(\partial\Omega_{2,T})} \leq C_T\|G\|_{L^2(\partial\Omega_{2,T})}.$$

Proof : To solve (72) in a unique way, it is sufficient that $\mathbb{T}$ is a contraction on $L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$ (or equivalently on $L^2(]-\infty, T] \times \mathbb{R}_+)$. A simple computation shows that is it effectively the case under the assumption $|S| < \sqrt{\beta}$.

□

Remark The fact that we are interested in uniform bounds (compared with the parameter γ) of the operator $\mathbb{T}$ is motivated by the following fact. In the analysis of the initial boundary value problem in the half space, one starts to deal with global problems in time. Then from the uniformity of the energy estimate compared to γ follows a principle of causality which allows to restrict to problems where the time variable lies in $]-\infty, T]$. We refer to [3] and [4] for more details about this proof.

To fully understand the condition $|S| < \sqrt{\beta}$ it is important to remark the following fact : if one considers a point $(0, L) \in \partial\Omega_1$ and follows the characteristic curves associated with the indeces on the loop, then after three reflections, this traveling point goes back to $\partial\Omega_1$ in a new position $(0, L') \in \partial\Omega_1$. Some basic computations show that :
$$\beta = \frac{L}{L'}. \tag{74}$$
So, we have three possible behaviours depending of the value of β :
◇ If $\beta > 1$, then traveling along the bicharacteristics the information approaches the corner.
◇ If $\beta < 1$, then traveling along the bicharacteristics the information goes away from the corner.
◇ If $\beta = 1$, then the travel along the bicharacteristics is periodic.
The condition $|S| < \sqrt{\beta}$ imposes that after one turn along the bicaracteristics associated with the loop the L^2-norm of the trace has descreased.

In the scalar case, that is when the matrix S can be replaced by the scalar $\widetilde{S}$,[4], that is to say when the rank of the projector $P_1^{n_1}$ is one, we can show that the condition $|\widetilde{S}| < \sqrt{\beta}$ is also necessary for the well-posedness of (71). The idea of the proof is to use Laplace transformation in the time variable to reduce to a situation already studied by Osher in [16].

Theorem 4.4 *Let* $\alpha, \beta > 0$ *and* $\widetilde{S} \in \mathbb{R} \setminus \{0\}$ *be such that* $\widetilde{S} > \sqrt{\beta}$*. Then the equation :*
$$u(t, x) - \widetilde{S}u(t - \alpha x, \beta x) = G, \tag{75}$$
satisfies one of the alternatives : i) if $\beta < 1$*, then equation* (75) *written for* $G = 0$ *admits a non-zero solution in* $L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$*, for all* $\gamma > 0$*.*
ii) If $\beta > 1$*, then there exists* $g \in L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$ *such that equation* (75) *does not have any solution.*

[4]When S is a matrix and not a number, it seems more diffiult to show the analog of Theorem 4.4. That is why the restriction to stricly hyperbolic operators is a easy way to obtain sharp results.

Proof : We begin with the proof of i). We are looking for a non-zero solution u written under the form $u(t,x) = H(t)v(t,x)$, where H is the Heaviside function. Applying Laplace transform in the time variable to equation (75), lead us to solve :

$$\hat{v}(\sigma, x) - \widetilde{S}e^{-\alpha\sigma x}\hat{v}(\sigma, \beta x) = 0, \tag{76}$$

where $\sigma \in \mathbb{C}$, Re $\sigma > 0$, is the dual variable of t. Following [16], let

$$\hat{\tilde{v}}(\sigma, x) = e^{\frac{\alpha\sigma x}{1-\beta}} x^{-\frac{\ln \widetilde{S}}{\ln \beta}}.$$

It easy to check that this function is a solution to (76). But Re $\left(\frac{\alpha\sigma}{1-\beta}\right) < 0$, so using the assumption $\widetilde{S} > \sqrt{\beta}$, it follows that $\hat{\tilde{v}} \in L^2_x(\mathbb{R}_+)$.

However, in view to come back to the time variable, we want to apply the Paley-Wiener Theorem to $\hat{\tilde{v}}$. That is why we denote by $\hat{v}(\sigma, x)$ the following modification of $\hat{\tilde{v}}$:

$$\hat{v}(\sigma, x) = \frac{1}{(1+\sigma)}\hat{\tilde{v}}(\sigma, x).$$

It is easy to see that $\hat{v}$ is still a solution to (76). Moreover

$$\sup_{\gamma>0}\int_{\mathbb{R}}\int_{\mathbb{R}_+} |v(\gamma + i\eta, x)|^2 dx d\eta \le \sup_{\gamma>0}\left(\int_{\mathbb{R}_+} x^{-2\frac{\ln \widetilde{S}}{\ln \beta}} e^{\frac{2x\alpha\gamma}{1-\beta}} dx\right)\int_{\mathbb{R}} \frac{1}{1+\eta^2} d\eta \le C.$$

We can thus apply Paley-Wiener Theorem, so there exists $v \in \bigcap_{\gamma>0} L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)$ such that $\hat{v}$ is the Laplace transform. As a consequence we have constructed a non-zero solution to (75).

To show ii), using the same proof as in [16], it is sufficient to remark that the adjoint of $\mathbb{T}$ is given by :

$$\mathbb{T}^* v = -\frac{\widetilde{S}}{\beta} v\left(t - \frac{\alpha}{\beta}x, \frac{1}{\beta}x\right).$$

So if $\beta 11$, the operator $\mathbb{T}^*$ is in the situation i), so $\frac{\widetilde{S}}{\beta} > \frac{1}{\sqrt{\beta}} \Leftrightarrow \widetilde{S} > \sqrt{\beta}$, and $\mathbb{T}^*$ is not injective. As a consequence, $\mathbb{T}^*$ is not surjective.

□

During the construction of the geometric optics expansion we have seen that the invertibility of the operator $I - \mathbb{T}$ on the weighted space $L^2_\gamma(\mathbb{R}\times\mathbb{R}_+)$ was not sufficient to construct the term of order ε which is however necessary if we want to show that the truncated expansion approximates the exact solution. More precisely to construct the first corrector term it is necessary that $I - \mathbb{T}$ is (at least) invertible from H^3_f to H^3_f, and more generally if one wants that the remainder $u^\varepsilon - u^\varepsilon_{app,N_0}$ be $O(\varepsilon^{N_0+1})$, it is needed that $I - \mathbb{T}$ is invertible from $H^{N_0+3}_f$ in $H^{N_0+3}_f$ (to ensure that the term of order $N_0 + 1$ is at least in $H^1(\Omega_T)$. The following Theorem shows that the solution to the functional equation (71) given by Theorem 4.3 inherits (under some restrictions) the regularity H^K_f of the source term. There are two different cases to handle with :

Theorem 4.5 *i) If $0 < \beta \le 1$ and if $|S| < \sqrt{\beta}$, then $I - \mathbb{T}$ is invertible from H^∞_f to H^∞_f.*

ii) Let $K \in \mathbb{N}$, if $\beta > 1$ and if $|S|\beta^{K-\frac{1}{2}} < 1$, then $I - \mathbb{T}$ is invertible from H^K_f to H^K_f.

Proof : Let

$$u(t, x_1) = G(t, x_1) + \sum_{j=1}^{+\infty} S^j G\left(t + X^j_{\alpha,\beta}x_1, \beta^j x_1\right), \tag{77}$$

where

$$X^j_{\alpha,\beta} := \sum_{k=0}^{j-1} \alpha\beta^k + \beta^j\alpha.$$

It is easy to check that, under assumption $|S| < \sqrt{\beta}$, u is a solution to (71) which belongs to $L^2(\partial\Omega_{2,T})$. According to Theorem 4.3, it is unique.

Then, we show that the solution u defined by (77) inherits the regularity of G. Firstly, according to the particular form of equation (71), it is clear that independently of β, for all $n \in \mathbb{N}$:

$$\|\partial_t^n u\|_{L^2(\partial\Omega_{2,T})} \leq C \|\partial_t^n g\|_{L^2(\partial\Omega_{2,T})}, \tag{78}$$

so we only have to deal with the derivatives in the spatial variable, because it will also permit to deal with the cross-derivatives by using (78). For $n \in \mathbb{N}$, a simple computation gives :

$$\partial_{x_1}^n u = \partial_{x_1}^n G + \sum_{j=1}^{+\infty} S^j \left[\sum_{l=0}^{n} \binom{n}{l} \left(X_{\alpha,\beta}^j\right)^l \left(\beta^j \kappa\right)^{n-l} \partial_t^{n-l} \partial_{x_1}^l G \right] \left(t + X_{\alpha,\beta}^j x_1, \beta^j x_1\right), \tag{79}$$

and lead us to a distinction depending on the value of β.

If $\beta \leq 1$, then all the constants appearing during the derivation can be abruptly bounded from above and we obtain :

$$\|\partial_{x_1}^n u\|_{L^2(\partial\Omega_{2,T})} \leq \|\partial_{x_1}^n G\|_{L^2(\partial\Omega_{2,T})} + C_{n,\alpha} \sum_{j=1}^{+\infty} \left(\frac{|S|}{\sqrt{\beta}}\right)^j \sum_{l=0}^{n} \|\partial_t^{n-l} \partial_{x_1}^l G\|_{L^2(\partial\Omega_{2,T})},$$

where we used the change of variable

$$\begin{bmatrix} s \\ y \end{bmatrix} = \begin{bmatrix} 1 & X_{\alpha,\beta}^j \\ 0 & \beta^j \end{bmatrix} \begin{bmatrix} t \\ x_1 \end{bmatrix},$$

to force the appearance of the factor $\sqrt{\beta}$. As a consequence, under assumption $|S| < \sqrt{\beta}$, the solution u given by (77) is $H^\infty(\partial\Omega_{2,T})$ and we can check on the equation (79) that its trace is also in H_f^∞.

If $\beta > 1$, we have :

$$\|\partial_{x_1}^n u\|_{L^2(\partial\Omega_{2,T})} \leq \|\partial_{x_1}^n G\|_{L^2(\partial\Omega_{2,T})} + C_{n,\alpha} \sum_{j=1}^{+\infty} \left(|S|\beta^{n-\frac{1}{2}}\right)^j \sum_{l=0}^{n} \|\partial_t^{n-l} \partial_{x_1}^l G\|_{L^2(\partial\Omega_{2,T})},$$

for $0 \leq n \leq K$, this sum is finite thanks to assumption $|S|\beta^{K-\frac{1}{2}} < 1$. We have thus shown that $u \in H^K(\partial\Omega_{2,T})$. Once again, the flatness at the corner is given by computing the trace in (79), so we have $u \in H_f^K$.

□

Remark As in the situation where an infinite number of phases was present in the WKB expansion (see example in paragraph 3.4), we can remark that when the source term $g \in \mathscr{C}_c^\infty$ with its support away from the corner, if we restrict ourselves to the construction of the WKB expansion for a finite time $T < +\infty$, then the number of non-zero terms in the sum (77) is finite. Thus, in this framework the operator $(I - \mathbb{T})$ is automatically invertible (independently of the parameters β and S). Its inverse is given by (77).

Theorem 4.5 seems to indicate that a corner concentration phenomenon is more difficult to handle with than a separation from the corner phenomenon. Indeed, if $\beta < 1$, the error between the exact solution and the truncated WKB expansion is $O(\varepsilon^N)$ with N arbitrarily large, whereas if $\beta > 1$, the norms of the derivatives of the solution to (71) seem to get larger and larger. To prevent this "blow up", we have made the assumption $|S|\beta^{K-\frac{1}{2}} < 1$ which "implies" that there exists a maximal N_0 such that the error is $O(\varepsilon^{N_0})$. We do not claim that, for $\beta > 1$, Theorem 4.5 is sharp. But it is sufficient to treat the example of paragraph 3.5.

4.5 Examples for which the invertibility condition (71) is satisfied.

4.5.1 The example of paragraph 3.5.

We come back to the corner problem (16) and more precisely on the resolution of the amplitude equation (71) for this problem.

First of all we have to specify the chosen boundary conditions. We set for B_1 and B_2 in (16) :

$$B_1 := \begin{bmatrix} 0 & 4 & \sqrt{7} \\ \sqrt{7} & -4 & 0 \end{bmatrix}, \; B_2 := \begin{bmatrix} -\delta & 1 & 0 \end{bmatrix}, \tag{80}$$

where $\delta \in \mathbb{R}$, $\delta \neq 0$ is a fixed parameter.

It is easy to see that the boundary condition defined by B_1 on the side $\partial\Omega_1$ is strictly dissipative, in particular it satisfies the uniform Kreiss-Lopatinskii condition (see [3, Proposition 4.4]). The real parameter $\delta \neq 0$ encodes the dissipativity on the side $\partial\Omega_2$ in the following way :

◇ If $|\delta| > 1$, the boundary condition defined by B_2 is strictly dissipative.

◇ If $|\delta| = 1$, the boundary condition defined by B_2 is maximal dissipative.

◇ If $0 < |\delta| < 1$, the boundary condition defined by B_2 is not dissipative but it satisfies the uniform Kreiss-Lopatinskii condition.

Reiterating the same kind of computations as those described in paragraph 4.2.2, using the fact that $\dim \mathscr{L}(d\varphi_1) = 1$, show that the amplitude equation for the amplitude associated with the phase φ_1 is scalar and reads :

$$u(t, 0, x_2) = -\frac{\sqrt{5}}{64\sqrt{7}\delta} \left[1 - 3\sqrt{5}\right] \left[\frac{7}{3} - \sqrt{5}\right] u\left(t - 27x_2, 0, \frac{35}{2}x_2\right) + G(t, x_2), \tag{81}$$

where, u is a real-valued function and for an explicitely computable, but non-relevant, source term G.

According to Theorem 4.3, if

$$|\delta| > \frac{1}{32 \cdot \sqrt{5} \cdot 7^{3/2}}(-1 + 3\sqrt{5}) \left(\frac{7}{3} - \sqrt{5}\right) := \delta_0 \approx 4.10^{-4}$$

then the functional equation (81) admits a unique solution. We are thus able to construct the leading order term of the geometric optics expansion for more parameters than those leading to strictly dissipative boundary conditions.

A contrario, if

$$0 < |\delta| < \delta_0,$$

we are in a non-dissipative framework, and according to Theorem 4.4, equation (81) admits a non-zero solution for $G = 0$, so the leading order term in the WKB expansion is not determined in a unique way. This example shows that imposing the uniform Kreiss-Lopatinskii condition on each side of the boundary is not sufficient to construct the geometric optics expansion in a unique way. It seems to be a good argument in favour of the fact that the same situation is true for the strong well-posedness of the corner problem (18).

We know that the corner problem (16) is strongly well-posed for $|\delta| > 1$ since the boundary conditions are strictly dissipative. To show that the truncated WKB expansion is a good approximation to the exact solution, as $\beta > 1$, we have to study the condition $|S|\beta^{K-\frac{1}{2}} < 1$. A simple computation shows that this condition is satisfied as long as $|\delta| > 1$ and $K \leq 4$. Thus, we can construct the geometric optics expansion up to the order ε^2 and Theorem 4.2 says that the truncated geometric optics expansion approximates the exact solution to (16), with an error in $O(\varepsilon)$, for all parameters δ making the boundary conditions of (16) strictly dissipative.

4.5.2 The example of Sarason-Smoller [17].

In [17], the authors construct an example of 4×4 a strictly hyperbolic operator whose section of the characteristic variety contains a loop. This example, with the example of paragraph 3.5 constitute, to our knowledge, the only two examples of corner problems with a loop in the literature.

The main idea of the construction is a perturbation argument : we first choose a centered ellipse and we fix three points P_2, P_3, P_4 on this ellipse such that angle $\widehat{P_2P_3P_4}$ be a perpendicular angle and the group velocities are incoming-outgoing in P_3 and outgoing-incoming in P_2 and P_4. This choice determines in a unique way a point P_1, such that $P_1P_2P_3P_4$ is a rectangle. Then we construct a second ellipse meeting P_1 with an incoming-outgoing group velocity at this point (see figure 8). The variety constructed can be written

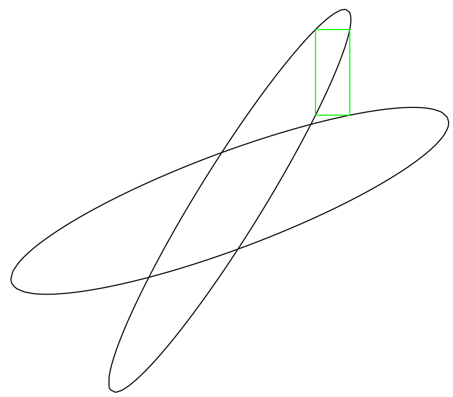

Figure 8: The construction of Sarason and Smoller for $p_1 := 5\xi_1^2 + 2\xi_2^2 - 6\xi_1\xi_2 - 1$ and $p_2 := \frac{25}{49}\xi_1^2 + \frac{14}{5}\xi_2^2 - 2\xi_1\xi_2 - 1$.

under the form :

$$p_1(\tau, \xi_1, \xi_2)p_2(\tau, \xi_1, \xi_2) = 0,$$

where the polynomials p_1 and p_2 are homogeneous of degree two. This variety contains the loop (P_1, P_2, P_3, P_4), but it can not represent the section at $\tau = 1$ of the characteristic variety of a strictly hyperbolic operator. Indeed, the two ellipses constructed previously intersect in four points, namely Q_1, Q_2, Q_3 and Q_4. However, it can be shown that it is the section at $\tau = 1$ of the characteristic variety of a geometrically regular hyperbolic operator with A_1 and A_2 of block diagonal form :

$$A_1 := \begin{bmatrix} -a_1 & a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & -\widetilde{a}_1 & \widetilde{a}_2 \\ 0 & 0 & \widetilde{a}_2 & \widetilde{a}_1 \end{bmatrix}, \text{ and } A_2 := \begin{bmatrix} -b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -\widetilde{b} & 0 \\ 0 & 0 & 0 & \widetilde{b} \end{bmatrix},$$

for suitable real parameters $a_1, a_2, \widetilde{a}_1, \widetilde{a}_2, b$ and $\widetilde{b}$ (we refer to [2, paragraph 6.9.6] or [17] for more details about the construction of A_1 and A_2).

Once the operator $L(\partial)$ is constructed, we add the following boundary conditions :

$$B_1 u_{|x_1=0} := g^\varepsilon, \quad B_2 u_{|x_2=0} := 0,$$

where B_1 and B_2 are defined by :

$$B_1 := \begin{bmatrix} 1 & 0 & 0 & -\delta \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 0 & 1 \end{bmatrix}. \tag{82}$$

It is easy to check, using the particular form of A_1 and A_2 and the fact that the boundary conditions (82) written for $\delta = 0$ are strictly dissipative, that B_1 and B_2 satisfy the uniform Kreiss-Lopatinskii condition for all δ. It is also easy to check that the boundary conditions B_1 and B_2 are strictly dissipative if δ is sufficiently small.

Now that boundary conditions are fixed, we want to study the invertibility condition $|S| < \sqrt{\beta}$, which appears when we construct the WKB expansion of the corner problem [5] :

$$\begin{cases} L(\partial)u^\varepsilon = 0, \\ B_1 u^\varepsilon_{|x_1=0} = g^\varepsilon, \\ B_2 u^\varepsilon_{|x_2=0} = 0, \\ u^\varepsilon_{|t\leq 0} = 0, \end{cases} \tag{83}$$

where the source term g^ε reads :

$$g^\varepsilon = e^{\frac{i}{\varepsilon}(t+P_{1,2}x_2)}g,$$

with $g \in H^\infty_f$, zero for negative times, and where $P_1 := (P_{1,1}, P_{1,2})$.

The factor β only depends of the coefficients of the operator $L(\partial)$. In particular, it does not depend of δ and can be explicitely computed. The term S can be considered as a scalar (see paragraph 4.4) and it is given by :

$$S := S_1^{P_1} B_1 S_2^{P_2} B_2 S_1^{P_3} B_1 S_2^{P_4} B_2,$$

where we have made the amalgam between the indeces of the loop and the associated frequencies.

Once again, using the fact that the operator $L(\partial)$ defined two 2×2 uncoupled systems, the stable subspace $E_2^s(i, P_{4,1})$ reads $E_2^s(i, P_{4,1}) = vect\,\{(0, 0, p_4, q_4), (\widetilde{p}, \widetilde{q}, 0, 0)\}$. Thus, we can easily compute :

$$S_2^{P_4} B_2 \begin{bmatrix} p_1 \\ q_1 \\ 0 \\ 0 \end{bmatrix} = \mathbb{P}^{P_4} \left(\begin{bmatrix} \frac{\widetilde{p}}{\widetilde{q}} q_1 \\ q_1 \\ 0 \\ 0 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 0 \\ \frac{p_4}{q_4}\left(q_1 \frac{\widetilde{p}}{\widetilde{q}} - p_1\right) \\ q_1 \frac{\widetilde{p}}{\widetilde{q}} - p_1 \end{bmatrix} \right) := \tilde{c}\delta \begin{bmatrix} 0 \\ 0 \\ p_4 \\ q_4 \end{bmatrix},$$

where $\tilde{c}$ is not zero and only depends of the projector $\mathbb{P}^{P_4}$ on $\ker(\mathscr{L}(\underline{\tau}, P_4))$ and where we set $\ker(\mathscr{L}(\underline{\tau}, P_1)) = vect\,\{(p_1, q_1, 0, 0)\}$. Repeating exactly the same arguments for the other terms composing S, it is easy to show that the invertibility condition (71) it equivalent to :

$$c\delta^2 < \sqrt{\beta}, \quad c > 0.$$

This condition is not satisfied for large values of δ. Let us remark the following points : firstly, the blow up phenomeon is δ^2 and not in δ^4 as predicted in [17]. Secondly, for $|\delta|$ small enough, the invertibility condition is satisfied and we are in the strictly dissipative framework.
Moreover, the condition $c\delta^2 < \sqrt{\beta}$ is more precise than the analogous condition of [17]. Indeed it says that since we are working with L^2-norms, to prevent the signal to increase in strenght after a complete circuit it had to be asked that the amplification caused by the boundary condition is less than the contraction of the support of the source term after a complete circuit.

Finally, since we have $\beta > 1$ and S "scalar", Theorem 4.4 tell us that if the condition $c\delta^2 < \sqrt{\beta}$ is not satisfied then the WKB expansion can not be constrcuted for ant source term on the boundary. This conclusion is, in fact, worst than the conclusion of Sarason and Smoller which says that in this case the corner problem is poorly-posed (see [17] Definition 4.3).

To construct a strictly hyperbolic corner problem whose characteristic variety contains a loop, Sarason and Smoller use a perturbation argument. More precisely, they introduce a small coupling between the two

[5] We do not try here to determine all the phases in the WKB expansion because, due to a concentration of the phases in the neighborhood of the intersection points of the ellipses (see paragraph 3.5), the number of expected phases in the expansion will be infinite.

2×2 uncoupled systems defining $L(\partial)$ constructed in such a way to "pull apart" the two intersecting ellipses of the characteristic variety see sections 7 and 9 of [17]. They show that if the perturbation is small enough then the obtained corner problem is strictly hyperbolic and the boundary conditions defined by B_1 and B_2 satisfy the uniform Kreiss-Lopatinskii condition, and finally that the perturbed system admits a loop in the section of its characteristic variety.

5 Conclusion.

In this article we have shown a Theorem which gives a rigorous geometric optics expansion for an hyperbolic corner problem when the number of phases generated by reflections is finite, but our Theorem is general enough to apply to problems involving selfinteracting phases. For such problems, a sequence of propagation of, at least, four fixed group velocities is repeated *ad vitam æternam.* The construction of the geometric optics expansion then needs the solvability of a new amplitude equation, which is an invertibility condition in the spirit of Osher's condition [14](cf. Assumption 4.2).

Of course, the construction given in this article can also be made if the section of characteristic variety does not contain any loop. Without any surprise, in that framework the construction is much more simpler, because the results in paragraphs 4.2.1 and 4.2.4 are more or less immediate and we can construct the expansion as it has been done in [2, Paragraphs 6.4-6.6]. Moreover, the construction can also be adapted if the source term on the boundary does not turn on a selfinteracting phase but if this phase appears after several reflections.

We also think that is should be possible to show a version of Theorem 4.2 without the assumption of non-appearance of glancing modes during the phase generation process. Indeed, if one starts with a hyperbolic frequency then nothing ensures that after several reflections a glancing mode will not appear in the phase generation process. However, thanks to [19]-[20], we think that, after the suitable modifications of the oscillating scales in the ansatz, glancing modes will, more or less, behave like evanescent modes in the sense that they will create boundary layers in the expansion and that they will not be reflected from one side to the other.

At last, the proof of Theorem 4.2 should also work when there are several loops. There are two cases to distinguish, first if there is still a unique loop of interaction but others loops that are not interaction loops and, secondly, the case where there are more than one interaction loops. In the first case, the proof of Theorem 4.2 has just to be a bit adaptated in a more technical way. In the second case, we think that the proof of Theorem 4.2 can also be adapted as long as the interaction loops do not intersect themselves but this is left for future work.

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